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ABSTRACT

This manual was written to help district curriculum leaders in New Jersey improve instruction in mathematical problem solving in middle and junior high schools. Four training modules contain the material and information needed to conduct a staff development course. The manual was designed to accompany turn-key training sessions to prepare the leader to conduct in-service training for teachers. The first section contains specific supervisory strategies for promoting effective problem-solving instruction. Section 2 provides general information for the curriculum leader to use in organizing in-service training based on the materials in this manual. Each of the four training modules in Section 3 contains a ready-to-reproduce workshop packet and a guide to the presenter which includes training materials. Module I introduces the problem-solving process and the need to improve problem-solving instruction. Module II focuses on the identification and teaching of effective problem-solving strategies and skills. The activities in Module III lead participants in analyzing the textbook treatment of problem solving and in expanding word problem-solving experiences beyond the textbook. Problem solving using models and manipulative materials is explored in Module IV. Each module describes training activities which incorporate implications from research, training tips, instructional materials, and follow-up activities. (MNS)

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# IMPROVING MATHEMATICAL PROBLEM-SOLVING SKILLS IN THE MIDDLE GRADES

## A Staff Training Manual

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**NEW JERSEY STATE DEPARTMENT OF EDUCATION**

# IMPROVING MATHEMATICAL PROBLEM-SOLVING SKILLS IN THE MIDDLE GRADES

A Staff Training Manual

Saul Cooperman
Commissioner of Education

Division of Compensatory/Bilingual Education
Sylvia M. Roberts, Director

New Jersey Department of Education
225 West State Street
Trenton, New Jersey 08625

Spring, 1985

PTM #400.60

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# INTRODUCTION

This manual was written to help district curriculum leaders improve instruction in mathematical problem solving in the middle and junior high schools. Four training modules contain the material and information needed to conduct a staff development course in mathematical problem solving. The manual was designed to accompany turn-key training sessions to prepare the leader to conduct in-service training for teachers.

The following are objectives for local leaders who participate in the accompanying turn-key training:

1) To explore the implications of research for the improvement of mathematical problem solving;

2) To increase awareness and understanding of appropriate mathematical problem-solving strategies and skills;

3) To identify appropriate problem-solving materials and resources;

4) To improve the ability to model good questioning techniques and teaching strategies for problem-solving instruction;

5) To develop a familiarity with the included in-service training materials; and

6) To examine supervisory strategies for the improvement of problem-solving instruction.

## Overview of the Manual

The first section of the manual contains specific supervisory strategies for promoting effective problem-solving instruction. The role of the district curriculum leader is crucial in providing the support, ideas, and environment to foster professional growth among teachers.

Section 2 provides general information for the curriculum leader to use in organizing in-service training based on the materials in this manual. Suggestions related to scheduling and preparation are included.

Each of the four training modules in Section 3 contains a ready-to-reproduce workshop packet and a guide to the presenter which includes training materials. Each module is separately paginated with a Roman numeral to identify the module and an Arabic numeral to indicate the sequence within the module. The workshop packet appears first so that it can be reproduced and appropriate pagination be maintained.

Module I introduces the problem-solving process and the need for improving instruction in mathematical problem solving. Module II focuses on the identification and teaching of effective problem-solving strategies and skills. In Module III, the activities lead participants

in analyzing the textbook treatment of problem solving and in expanding word problem-solving experiences beyond the textbook. Problem solving using models and manipulative materials is explored in Module IV. The content of the modules is interrelated to form a coherent course for teachers in the middle grades.

Each module describes detailed training activities which incorporate implications from research, training tips specific to the content of the module, and instructional materials and programs appropriate to the content. The modules include follow-up activities for participants to do with their students. The structure of the training sessions ensures an opportunity for the presenter to gather feedback about the implementation of the ideas presented in the workshop.

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# SECTION I

# THE ROLE OF THE SUPERVISOR IN PROMOTING EFFECTIVE PROBLEM-SOLVING INSTRUCTION

## THE ROLE OF THE SUPERVISOR

Effective supervision in any instructional area requires good communication between the teacher and supervisor. Whatever you can do to help teachers feel free to share their struggles as well as successes with you and each other will have a positive effect on problem-solving instruction.

Specific supervisory strategies for the improvement of problem-solving instruction are included in this section. These suggestions are addressed in three categories--modeling effective instructional techniques, structuring inter-staff sharing, and fostering staff development activities.

### Modeling Effective Instructional Techniques

**Demonstrate problem-solving instructional techniques with students.** Lead students in whole classes and small groups to solve problems. If you videotape your lessons, other teachers will be able to view them. Discuss your teaching with the staff in a self-critical way. Showing your willingness to analyze your teaching behavior will encourage staff to examine their own behavior.

**Interact informally with students during problem solving.** When you visit a class during problem-solving instruction, assist students individually. Ask them to verbalize their thinking strategies. Use questioning techniques to help them overcome difficulties.

**Assist teachers in the classroom to diagnose student difficulties in problem solving.** Show teachers how to use written work samples and oral interviews to assess a student's strengths and needs in mathematical problem solving. An audiotape of a diagnostic interview with a student can be shared with the staff. Your specific instructional suggestions to help the student will be appreciated by the teacher.

**Facilitate the development of bulletin boards and newsletters with a problem-solving focus.** Suggest problem solving as a focus for bulletin boards or newsletters. Some possible themes might be: student-generated story problems, contests for solving brain-teasers, mathematical problem solving in the world of work, problem solving and the computer, and calculator explorations of large numbers. Older students could be encouraged to write a problem-solving newsletter for younger students.

**Organize trips that highlight the problem-solving process in the world of work.** Trips to local merchants--grocers, butchers, record stores--can provide real-life applications of mathematical problem-solving skills. In work places such as automobile factories, airports, printing plants, computer companies, insurance offices, and utilities companies, students can have the opportunity to see adults using mathematical skills on the job.

**Prepare a problem-solving newsletter to be sent home with students.** Include student-authored problems and challenging problems that can be solved cooperatively by the family. FAMILY MATH is an example of a program designed to help parents work with their children in math. Request materials from: FAMILY MATH/EQUALS, Lawrence Hall of Science, University of California, Berkeley, California 94720.

Structuring Inter-staff Sharing

**Schedule common planning time.** If preparation periods are scheduled to increase time for interaction, teachers can meet together about instructional improvement in mathematics. Basic skills improvement program teachers need to communicate with teachers in the developmental program. Teachers in other content areas such as science and social studies need to discuss problem-solving instruction with the mathematics teachers.

**Encourage problem-solving activities in all content areas.** Every teacher is a teacher of problem solving. Many of the same strategies and skills appropriate to mathematical problem solving are applicable in other contexts. Supervisors in other content areas need to assist teachers in the development of problem-solving skills and strategies in their classrooms. Choose problems and activities that are appropriate to other content areas and share them.

**Establish a professional library in each school building.** Choose a secure, but accessible place for a professional library. Subscriptions to The Arithmetic Teacher and The Mathematics Teacher should be ordered for each school from the National Council of Teachers of Mathematics, 1906 Association Drive, Reston, Virginia 22091. Bulk orders of the journals make it possible for each teacher to have his or her own copy inexpensively. The reference lists that are included in each workshop packet in this manual suggest other books and journals that could be included in the school library. Collect catalogs from major publishers and manufacturers of mathematics materials. In addition, the library can serve as a repository for good problems and effective problem-solving lessons contributed by the staff.

**Disseminate copies of good problems, journal articles, and lessons.** Regularly review relevant publications. If you find a good problem-solving activity in the idea section of The Arithmetic Teacher, reproduce it for the teachers at the appropriate grade level. If you come across a challenging problem, distribute it. After attending a professional conference, share activities you gathered. Teachers will feel that you are thinking of their needs.

**Communicate resource needs to the district administration.** Act as a liaison between the teaching staff and the district administration. During the budgeting process, remind the administration to include needed problem-solving resources.

**Involve teachers in the curriculum alignment process.** Convene a committee to examine the mathematics curriculum in your district in terms of the demands of the New Jersey High School Proficiency Test and your local testing program. As the definition of basic mathematics skills is broadened, adjustments in the curriculum are needed.

## Fostering Staff Development Activities

**Arrange for staff to attend local, state, and national professional conferences and to share information with other staff members.** Teachers return from professional meetings with renewed enthusiasm. Try to arrange for as many teachers as possible to have the opportunity to attend conferences and meetings of professional associations such as the Association of Mathematics Teachers of New Jersey (AMTNJ) and the National Council of Teachers of Mathematics (NCTM). AMTNJ holds conferences each spring for teachers of elementary mathematics. NCTM holds conferences in the region every few years.

**Encourage staff to visit other districts, schools, and classes to observe problem-solving programs in action.** If you actively participate in professional associations, you will hear of model programs and new resources that would be of interest to the staff. When you send a staff member to check on the value of new materials, you are indicating your respect for that person.

**Periodically assess staff needs and interests related to improving problem-solving instruction.** Ask teachers to formally and informally articulate their staff development needs. A written needs assessment questionnaire can be designed to gather information which will be useful to you in planning staff development activities. Discuss ideas with staff members to informally learn about their interests and perceptions.

**Structure staff training opportunities that develop teachers' skills in the area of problem solving.** Staff development activities should be designed to foster improved instructional techniques. Some possible topics that could be addressed are: the nature of the problem-solving process, problem-solving strategies, cooperative problem solving, uses of test results and informal assesment to plan for instruction, activities that involve life applications, problem solving on the microcomputer, and activities that help the low-achieving student experience success in problem solving. In-service training is one way to encourage staff development. The subsequent sections of this manual will help you organize and conduct an intensive course in mathematical problem solving for teachers in the middle grades.

# SECTION 2

# ORGANIZING TO CONDUCT IN-SERVICE TRAINING

## ORGANIZING TO CONDUCT IN-SERVICE TRAINING

### Scheduling In-service Training

Although each of the four training modules included in this manual was designed to be used in approximately three hours, the actual time it will take a particular trainer to cover the material will vary greatly depending upon the style of the trainer and the degree of group participation. Suggestions are included in the presenter's guides to help the trainer pace the workshops.

To accommodate districts that choose to conduct the training in shorter time spans, each module contains several topic areas. Each module could be subdivided by these topics quite naturally to deliver effective training in a greater number of shorter sessions.

At least a week should pass between each training session to allow time for the participants to try out the content with their students. However, if too much time elapses between the sessions, continuity is difficult to maintain.

Consider the following in scheduling the in-service training:

- Consult the participants about dates and times that are most convenient for them;
- Provide incentives for teachers to participate, especially if the sessions are conducted after school hours;
- Sessions that are held during the regular school day allow for the demonstration of some of the activities with students;
- Leave enough time in the schedule for reproduction of materials and for ordering materials, if necessary;
- The location of training is important so make sure that a comfortable and appropriate place is available before the dates are set; and
- Remind participants about each training session a week before the date by distributing an agenda.

### Preparing the Training Materials and Environment

The first page of each presenter's guide contains a list which identifies what is needed for the training. The workshop packet for each module should be reproduced in its entirety for each participant. For some sessions, copies of various handouts must be prepared. Overhead transparencies should be made from the print copies.

Integrating local test results and samples of students work into the training would be beneficial. For example, in Module I, a written summary of district student performance could be prepared to add to the information on the National Assessment of Educational Progress. In addition, the included descriptions of student behavior could be replaced with district student work samples for the analysis of difficulties students encounter.

In Module IV, manipulative materials are needed. Cuisenaire Rods, Pattern Blocks, Tangrams, Geoboards, and Centimeter Cubes can be ordered from either of the following companies or another company that handles mathematics materials:

Creative Publications
3977 East Bayshore Road
P.O. Box 10328
Palo Alto, California 94303

Cuisenaire Company of America
12 Church Street
Box D
New Rochelle, New York 10805

Many districts already have these materials, particularly in primary classes in the elementary school. Paper versions of Pattern Blocks, Attribute Blocks and Tangrams can be reproduced from the training resources in Module IV.

Creating a comfortable training environment will contribute to the success of the workshops. Ideally, participants should sit around tables in small groups. Not only will such an arrangement facilitate communicaton, but also decrease loss of time as the workshop format shifts between large and small groups.

An overhead projector should be placed so that it can be easily viewed by all the participants. A chalkboard will be required for Workshop IV and will be useful in the other workshops as well. A tape recorder might be needed, if participants choose to record the feedback activities they will do with their classes.

Plan to provide refreshments for the participants and allow break time in the three-hour session.

## Reviewing for the Training

Even though you have probably attended the turn-key training workshops, you will need to reread the modules carefully to prepare yourself for conducting the training. The workshop packet and the presenter's guide with the training resources for each module should be studied simultaneously to gain an overall picture of the training content. All four modules are interrelated and should be digested before training begins.

Solve all the problems in the workshop packets in as many ways as you can. Generating alternate solutions for problems will help you anticipate the solutions workshop participants and students might use. Try to pose the problems to students at various levels of achievement. A tape recording of students verbalizing their thinking processes during problem solving would be extremely useful in the training sessions.

Plan the questions you will ask to stimulate discussion. Many suggestions are included in the presenter's guides. It is usually a good idea to be prepared with more questions than you will be able to use. Through the effective use of questioning in the training sessions, you will be modeling the kind of instructional techniques that are vital to the problem-solving process.

# SECTION 3

# TRAINING MODULES

MODULE I

# PROBLEM SOLVING - THE MOST BASIC MATHEMATICS SKILL

PACKET FOR WORKSHOP I

# Problem Solving - The Most Basic Mathematics Skill

New Jersey State Department of Education
Division of Compensatory/Bilingual Education
Spring, 1985

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## INTRODUCTION TO WORKSHOP I

Problem solving is a basic skill, or rather, is a basic process composed of many skills. The traditional focus of the elementary school curriculum has been on teaching arithmetic procedures and on practicing them. The teaching of mathematical problem solving has been neglected, even though the mastery of computational skills in isolation from problem solving is of limited use.

The "Position Paper on Basic Mathematical Skills" of the National Council of Supervisors of Mathematics (1977) was developed to encourage an expanded definition of basic skills beyond computation. The 10 basic skills areas from that position paper are listed below; the skills are interrelated and many of them have direct relevance to problem solving, the first on the list:

Problem solving;
Applying mathematics to everyday situations;
Alertness to the reasonableness of results;
Estimation and approximation;
Appropriate computational skills;
Geometry;
Measurement;
Reading, interpreting, and constructing tables, charts, and graphs;
Using mathematics to predict; and
Computer literacy.

An Agenda for Action (1980), presented by the National Council of Teachers of Mathematics (NCTM) as recommendations for mathematics education in the 1980s, further emphasized the need to make problem solving the focus of school mathematics. In addition, the recently released major national reports on the state of education call for improved curriculum and instruction in higher-level thinking skills and problem solving. In New Jersey, the new High School Proficiency Test requires students to solve multi-step problems and to apply computational skills.

What does it mean to make problem solving the focus of mathematics instruction?

It means:

- Teaching computational skills in the context of problem solving;
- Using a problem-solving approach to teaching computation, helping students understand the arithmetic algorithms they use;
- Weaving problem-solving instruction into the entire curriculum;
- Going beyond the textbook to create meaningful problem-solving situations;
- Teaching problem-solving strategies and skills;

- Helping students experience success in problem solving to motivate them to do more problem solving; and perhaps most of all,
- Modeling good problem-solving behavior for the students.

## OBJECTIVES FOR WORKSHOP I

The objectives of this workshop are:

1. To examine the nature of a problem and the problem-solving process;

2. To explore the skills needed for effective mathematical problem solving; and

3. To develop the ability to analyze student difficulties in mathematical problem solving.

## WHAT IS A PROBLEM?

### Activity I.1

Work with a partner to identify the phrases represented by each puzzle that is projected on the screen. Think about these questions as you figure the puzzles out:

What prerequisite information is needed?

What problem-solving strategies are you using?

What makes these puzzles problems for you?

### Activity I.2

Consider this situation:

> Lynette has 8 pennies, Alex has 12 pennies, and Alphonso has 5. How many do they have altogether?

If you ask a five-year-old to find the answer, he or she might create a collection of 8 pennies, of 12 pennies, and of 5 pennies together, and then carefully count to find the answer. This is a problem for a young child. To a twelve-year old, providing the answer to this situation is probably no problem. The older child has mastered an algorithm to find a quick, efficient solution, almost automatically.

What is a problem for one person may not be one for another. A situation becomes a problem when the context is new or unfamiliar for the problem solver.

Here is another statement about children and pennies:

> Barbara has some pennies. She has exactly three times as many as Al and exactly four times as many as Carlos. They have less than 100 pennies altogether. What can you say that is not obvious? How many pennies might each child have?

It is obvious that Barbara has more pennies than either Al or Carlos. Who has more, Al or Carlos? Does Barbara have more than Al and Carlos put together? Let us look more deeply into this situation. This is a problem for us, since we have no readily available algorithm or procedure to use to figure out how many pennies each might have.

Use the questions below to explore this problem. Talk to a partner about what makes this situation a problem.

Could Barbara have 7 pennies? _____ How about 11 pennies? _____
Why or why not? _____

What is the smallest number of pennies Barbara could have? _____ In that case, how many would the children have altogether? _____

What is the largest number of pennies Barbara could have? _____ So, Barbara must have _____ or _____ or _____ or _____ or _____ pennies.

Together the three children must have _____ or _____ or _____ or _____ or _____ pennies.

Activity I.3

Recall your solution to this problem:

> Barbara has some pennies. She has exactly 3 times as many as Al and exactly 4 times as many as Carlos. They have less than 100 pennies among them. How many pennies might each child have?

Record the steps you took in solving it.

Think about other ways to solve the same problem.

Activity I.4

List specific problem solving strategies and skills that you used in solving the problem.

Solve this problem. Add any additional problem solving skills or strategies that you use to the above list.

In my carpenter shop, I make three-legged stools and four-legged tables. One day I looked at my day's output and counted 31 legs. How many stools and how many tables might I have made?

Compare the carpenter shop problem with the pennies problem in terms of the format, the ease of solution, and the strategies and skills used.

## THE PROBLEM WITH PROBLEM SOLVING

### Activity I.5

Discuss in small groups the implications of the following results of the Second National Assessment of Educational Progess (NAEP) in mathematics:

- 56% of the thirteen-year-olds selected the correct answer to the following problem:

  > One rabbit eats 2 pounds of food a week. There are 52 weeks in a year. How much food will 5 rabbits eat in a week?

- 40% of the thirteen-year-olds correctly solved a word problem that requires dividing a 3-digit number by a 2-digit number; 70% could perform the division when given the same numbers in a standard computational format.

- About 85% of the thirteen-year-olds correctly responded to one-step word problems involving addition and subtraction.

- 54% of the thirteen-year-olds and 81% of the seventeen-year-olds solved this problem:

  > To make a cake, you need 2 1/4 cups of sugar for the frosting and 1 1/2 cups of sugar for the cake. How many cups of sugar are needed for the cake and frosting together?

- On addition exercises using fractions similar to those in the above problem, more than 46% of the thirteen-year-olds and 67% of the seventeen-year-olds calculated correctly.

- For most single-operation verbal problems, performance is closely related to computational skill. An exception involved multiplication of fractions. 70% of thirteen-year-olds correctly multiplied 2/3 x 2/5, but fewer then 20% could solve a verbal problem similar to the following:

  > Jane lives 2/3 mile from school. When she walked 2/5 of the way, how far had she walked?

- 11% of the thirteen-year-olds and 29% of the seventeen-year-olds solved this problem:

  > Lemonade costs 95¢ for one 56-ounce bottle. At the school fair, Bob sold cups holding 8 ounces for 20¢ each. How much money did the school make on each bottle?

- 40% of the thirteen-year-olds and 25% of the seventeen-year-olds took two of the numbers from the above problem and performed a single operation to arrive at their incorrect solution.

- More than half of the thirteen-year-olds and almost a third of the seventeen-year-olds could not identify what additional information would be needed to solve the following problem:

  > Lena left at noon to take a trip on her bicycle. She rode 5 miles each hour. Later that afternoon, Amanda decided to go after her. Amanda rode 10 miles each hour. What else would you need to know in order to find out how far the two girls rode before Amanda caught Lena?

- 50% of the thirteen-year-olds agreed with the statement, "Learning mathematics is mostly memorizing."

- 90% of the thirteen-year-olds agreed with the statement, "There is always a rule to follow in solving mathematics problems."

Activity I.6

Examine the following question from the mathematics section of the High School Proficiency Test administered in the Spring of 1984:

> The Johnson family made a 250-mile trip in their car. They left home at 8:00 a.m. and arrived at their destination at 2:40 p.m. the same day. If they took three 15-minute breaks to rest and a 50-minute break for lunch, how much time did they actually spend driving?

Generate a list of possible sources of error for students in solving this problem. What wrong answers might they give?

_____

_____

_____

_____

_____

_____

_____

Share your findings with the group.

Activity I.7

Form small groups. Evaluate the following work samples in terms of the student's skills and difficulties in problem solving. Discuss the instructional implications of what you learn about each student.

PROBLEM: Sixteen quarts of milk are supplied to the kindergarten children at Oak Street School each day. The milk is packaged in 8-ounce cartons, and all of it is consumed. If 18 of the children drink two cartons each during the day, and the rest drink one carton each, how many children attend kindergarten in the school?

Work Sample A

18 children drink 36 cartons
16 x 4 = 64 cartons total

36 64

Notes on Analysis of Work Sample A

Instructional Recommendations

PROBLEM: Sixteen quarts of milk are supplied to the kindergarten children at Oak Street School each day. The milk is packaged in 8-ounce cartons, and all of it is consumed. If 18 of the children drink two cartons each during the day, and the rest drink one carton each, how many children attend kindergarten in the school?

Work Sample B

Number of Cartons? = 64

18 drink 36 cartons

? drink 64 cartons

So $\frac{18}{1} \times \frac{64}{1} \times \frac{1}{36} = 32$

Notes on Analysis of Work Sample B

Instructional Recommendations

PROBLEM: Maria purchased 3 lbs of apples at $.69 per pound, 1 1/2 lbs of bananas at $.26 per pound, and 6 oranges at a price of 2 for $.79. How much did she spend altogether? (This problem is from the New Jersey High School Proficiency Test administered in the Spring of 1984.)

A. $7.20

B. $4.83

C. $4.04

D. $1.74

Work Sample C

[The student chose (A) as the answer.]

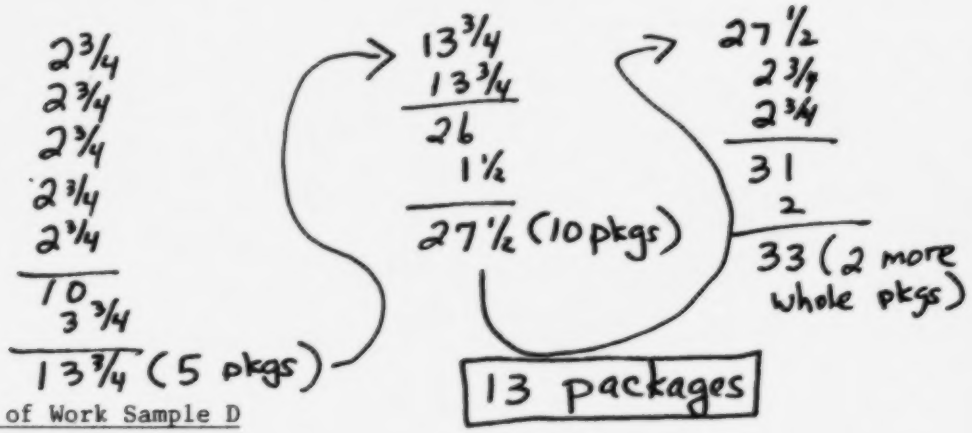
2.07
.39
4.74
7.20

Notes on Analysis of Work Sample C

Instructional Recommendations

PROBLEM: In the local grocery store, the butcher wraps ground beef in 2 3/4 pound packages. One day 35 pounds of ground beef were put into packages. How many packages were made?

Work Sample D




Notes on Analysis of Work Sample D

Instructional Recommendations

PROBLEM: A box of number two pencils contains 25 pencils and costs 87¢. How much will 75 pencils cost?

Work Sample E

Juan, a sixth grader, read the problem aloud and explained his solution attempt to the interviewer as follows:

Juan: "Um . . . ."

Interviewer: "Tell me what you are doing now."

Juan: "I'm trying to figure out how I'll do this . . . You either add or multiply . . . . O.K. I'll multiply the 75 pencils times 87¢. [multiplies on paper] It comes out to $65.25. No, that's not right."

Interviewer: "Why do you say that?"

Juan: "It's too much. Pencils don't cost that much."

Interviewer: "What are you trying to find in this problem?"

Juan: "The right answer."

Interviewer: "But what would lead you to the right answer?"

Juan: "What I need to do with 25, 87, and 75, because those are the numbers I need to use. [Silence as he rereads the problem.] Maybe I multiplied wrong. [He remultiplies, but is still stumped.]"

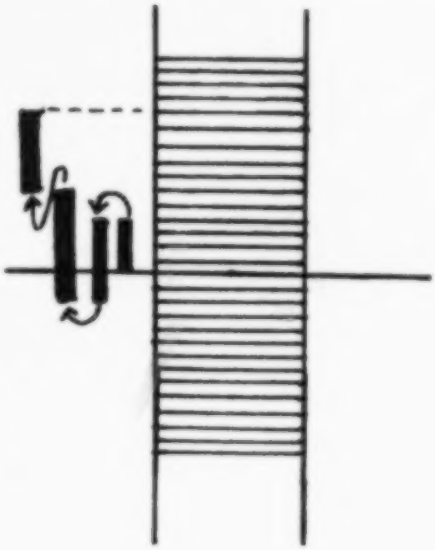
Notes on Analysis of Work Sample E

Instructional Recommendations

PROBLEM: A firefighter stood on the middle rung of a ladder as she directed water into a burning building. When the smoke lessened a little, she climbed up 4 rungs. A sudden gust of wind caused the fire to flare up, so she backed down 6 rungs. Later, she climbed up 8 rungs and stayed there until the fire was out. Then, she climbed the remaining 5 rungs to the top of the ladder and entered the building. How many rungs did the whole ladder have?

Work Sample F

11
x 2
22 rungs

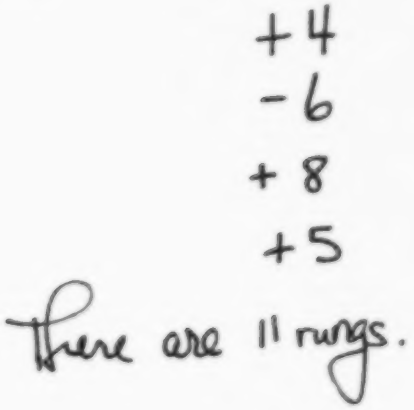


Notes on Analysis of Work Sample F

Instructional Recommendations

PROBLEM: A firefighter stood on the middle rung of a ladder as she directed water into a burning building. When the smoke lessened a little, she climbed up 4 rungs. A sudden gust of wind caused the fire to flare up, so she backed down 6 rungs. Later, she climbed up 8 rungs and stayed there until the fire was out. Then, she climbed the remaining 5 rungs to the top of the ladder and entered the building. How many rungs did the whole ladder have?

Work Sample G




Notes on Analysis of Work Sample G

Instructional Recommendations

Name _____

Date _____

## FOLLOW-UP ACTIVITY FOR WORKSHOP I

Choose one of the problems from this workshop or another multi-step problem to give to your students. It should be a problem for your students, that is, they should have the prerequisite knowledge, but should not be able to get an automatic, algorithmic solution without the need for analysis of the situation. Record the problem you use here:

You may want to introduce the problem orally to the group and help them explore it before they try to solve. Or, you may choose to assign the problem without introduction. Allow ample time for solution; this is not a test. Encourage the students to write all their work down on the page you will collect.

Analyze the written responses of the students. Record below any interesting observations about the written work.

_____

_____

_____

_____

_____

_____

_____

Choose two students to interview individually. Be sure to choose at least one who solved the problem incorrectly. Ask each to solve the same problem aloud, either by reviewing the written solution or by re-solving it. Ask probing questions and assist each student through questioning where help is needed. Record below a summary of the solution processes used.

_____

_____

_____

Think about the instructional implications of what you learn. Be prepared to share your findings with the group at the beginning of the next workshop.

If you tape record your interview, bring the tape of any interesting student responses to the next session.

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WORKSHOP PRESENTER'S GUIDE

Module I: Problem Solving - The Most Basic Mathematics Skill

## Trainer Overview of Module I

WORKSHOP OBJECTIVES:

1. To examine the nature of a problem and the problem-solving process;

2. To explore the skills needed for effective mathematical problem solving; and

3. To develop the ability to analyze student difficulties in mathematical problem solving.

TRAINING MATERIALS:

Workshop Packet I
Overhead Transparency A: Objectives for Workshop I
Overhead Transparency B: Puzzles
Overhead Transparency C: Piles of Counters
Overhead Transparency D: Pennies Problem
Overhead Transparency E: Definition
Overhead Transparency F: Carpenter Shop Problem
Overhead Transparency G: The Problem with Problem Solving
Overhead Transparency H: Problem from the 1984 High School Proficiency Test
Overhead Transparency I: Topics and General Contents of Each Session
Blank Transparencies
Chart Paper and Markers

PREPARATION SUGGESTIONS:

- Read the procedures and anticipate questions that may be asked by the participants.

- Solve all the problems in as many different ways as you can.

- Pose all the problems to individual students and have them think aloud as they solve them. You may want to tape record the interviews to use as illustrations in workshop sessions.

- Read some of the articles listed in the references section of the workshop packet. You might want to reproduce one of them to be distributed to participants after the workshop.

- Prepare the overhead transparencies and sufficient copies of the workshop packet.

## Procedures for Workshop I

### INTRODUCTION

Use Overhead Transparency A to introduce the objectives for the workshop. Indicate that this session is the first of four workshops. An overview of the remaining three sessions will be given at the end of this workshop.

### WHAT IS A PROBLEM?

#### Activity I.1

Project Overhead Transparency B or an alternate non-numerical problem. Games Magazine is a good source of appropriate puzzles for this activity.

Begin the workshop by asking the participants to identify the phrase each puzzle represents. Encourage them to work with a partner and to think aloud as they try to solve them.

Elicit answers, and more importantly, a description of their thinking during the solution process. Integrate the following points into the discussion:

- All prerequisite information was previously acquired. For example, everyone in the group could already read the words, recognize the shapes, and identify the relationships before attempting to solve the puzzles.
- What makes the puzzles problematic is the unfamiliarity of the context. The information must be put together in a new way.
- As more and more of these puzzles are solved, they become less and less problematic. For example, after "mind over matter" is solved, the context for "long underwear" is more familiar.

#### Activity I.2

Use the first situation listed in the workshop packet as an example of an appropriate mathematical problem for a preschool child, but not for a middle-school child. The discussion should cover the following issues:

- A preschool child can concretely solve the problem by manipulating objects and counting. Once introduced to the vocabulary of such a problem, the young child invents actions that will solve the problem. Such a situation can be solved automatically by most students in the upper elementary grades.

- Some situations can be a problem for the preschool child and the upper elementary child, but their methods of solution reflect their developmental levels. For example, consider the following problem on Overhead Transparency C:

  **Can 23 counters be put in piles so that each pile has three or four counters?**

- A five-year-old who can count to 23 can take 23 counters and try to put the counters in groups of three. When there are two counters left, the child might try to put one or more counters in each of two groups to show that the counters can be arranged in groups of three and four.

- An eight-year-old might deal with such a problem on a more representational level by drawing pictures of the objects and sketching the solution.

- An eleven-year-old might look at the problem and solve it mentally by using symbols or numerals:

  If I divide 23 by 3 that would give me 7 piles with a remainder of 2 counters, and one more for 2 of the piles. Yes, 23 counters can be put in piles so that each has three or four.

- For all three children the situation is a problem because there is not a standard algorithmic solution. All three children are using a trial and error strategy to approach the problem. The difference in their solution is in the mode of representation used to express the strategy; that is, there is a continuum from the concrete representation used by the five-year-old to the abstract expression of the strategy by the eleven-year-old.

Project Overhead Transparency D. Have participants work in pairs to solve the problem. Encourage them to think aloud as they try to solve the problem. Ask them to be aware of the problem-solving strategies they are using. Focus the attention on the problem-solving process rather than the answer.

Allow five to ten minutes for solving the problem. Use questions to assist them, only if they absolutely need help. If you need to stop them before they reach a solution, point out the frustration inherent in doing so. This suggests the importance of allowing adequate time for solving problems.

Ask participants to share their reactions to the problem-solving experience. The specific problem-solving strategies used will be the focus of Activity I.3. Discuss here what makes this situation a problem for them. For example, the problem has multiple answers; there is no way to set up algebraic equations; and there is no standard way of solving the problem. Integrate the following information into the discussion:

- Most textbook word problems are not really problems for the students. Instead, they are exercises which provide application situations for the computational skills that the students are developing. Many students do have difficulty in solving these exercises. However, developing better problem-solving skills can help students to complete these exercises more successfully. Using only application exercises to provide problem-solving practice for students stunts the development of problem-solving skills. Children learn early that they need not think about the word problems that appear in chapters on addition; they will do quite well on those questions, if they simply add the numbers which appear in each one.

- Teachers must provide non-routine problems--real problems in the sense we are discussing-if they hope to help students learn to persevere in the face of a puzzling situation, learn to approach situations flexibly, learn to analyze situations, in short, learn to become good problem solvers.

- Problem solving is a process that requires the application of skills, knowledge, and understandings which were previously acquired to a new set of conditions. Overhead Transparency E contains the definition of the problem-solving process used by the National Council of Supervisors of Mathematics in its "Position Paper on Basic Mathematical Skills."

## WHAT ARE THE SKILLS NEEDED FOR PROBLEM SOLVING?

### Activity I.3

Ask the participants to record in their workshop packets the steps they took in solving the problem about the children and the pennies from Activity I.2. Have several people report to the group how they solved it. Encourage them to be as specific as possible. You may want to lead each respondent step-by-step through the process used. Ask the group to generate other possible methods of solution.

Take notes on the responses on a blank overhead transparency, chart paper, or a chalkboard. Highlight some of the following points in your discussion:

- Problems need to be read several times before they are understood. Students have, unfortunately, become accustomed to instant answers in math.

- Verbalizing problem solving processes is a learning experience. It helps students become aware, to turn around and reflect on the techniques they have used.

- Problem-solving skills are developed through exposure to diverse problem-solving experiences.

- o There is no easy, one-to-one correspondence between words and operations. The "key word" approach often leads students to error. For example, in the pennies problem, the word "altogether" appears, but simple addition would not get an answer.

- o Perseverance and flexible thinking are crucial to the problem solving process.

Activity I.4

Ask participants to generate a list of specific problem-solving strategies and skills that were used in solving the pennies problem. Record their responses on an overhead transparency or chart paper. You will want to organize this list, and use it again at the next workshop.

As the participants suggest skills, you may want to help them see the skills as falling into the following categories: understanding the problem, devising a plan, carrying out the plan, and looking back. This taxonomy will be used in Module II to organize a more specific look at problem-solving strategies.

Present the carpenter shop problem using Overhead Transparency F. Suggest that the participants think about how this problem compares with the pennies problem as they attempt to solve it. The solution of this second problem will assist participants in identifying other problem-solving skills and strategies that can be added to the list. Discuss the two problems in relation to one another.

Some strategies and skills that might be suggested are listed below. You may want to point out that most of the strategies and skills that are important in mathematical problem solving are applicable to problem solving in general.

Trial and error;

Restating the problem;

Brainstorming information;

Identifying necessary data and assumptions;

Working backwards;

Making charts and tables;

Sketching diagrams and pictures;

Relating the problem to a familiar one;

Recognizing patterns;

Manipulating objects;

Using mental arithmetic;

Recognizing the reasonableness of an answer;

Making organized lists;

Breaking the problem down into subproblems;

Looking at a problem another way;

Generalizing; and

Representing information algebraically.

Each identified strategy and skill is clarified below as a resource for you. You may want to use these excerpts or descriptions as examples during the discussion. Most of the problems used appear in one of the training modules. You may want to refer to these examples in future training sessions.

## Trial and Error

A student solving the pennies problem might think:

> Let me make a guess. What if Barbara had 15 pennies? She has 3 times as many as Al so Al would have 5 pennies. She has 4 times as many as Carlos. Oh...15 can't be divided evenly by 4 so that can't be right. I need to make a guess for Barbara of a number that can be divided by 4. I'll try 60.

## Restating the Problem

The problem: "Hiroko is bigger than Suzanne. Suzanne is bigger than Jennie. Who is smallest?" might be restated by a student as follows:

> If Hiroko is bigger than Suzanne, then Suzanne is smaller than Hiroko. Jennie is smaller than Suzanne. Who is smallest?

## Brainstorming Information

A student might brainstorm information in solving ii as follows:

> Two eyes above the line, eyes over dots, eyes over two dots, two dots filled in dots, dots below eyes, dots under the eyes, circles, dark circles, dark circles under the eyes...

## Identifying Necessary Data and Assumptions

In solving, "How many squares are on the checkerboard?" a student might need to think about the following data and assumptions: the size of a checkerboard, the definition of a square, the different-sized squares on the checkerboard.

## Working Backwards

In the following problem, starting with the final outcome and working towards the beginning is an effective strategy:

> Jacob baked some cookies. He put half of them away for the next day. He ate 7 of them. Then, he divided the remaining cookies evenly among his three sisters so that each got 4. How many cookies did he bake?

## Making Charts and Tables

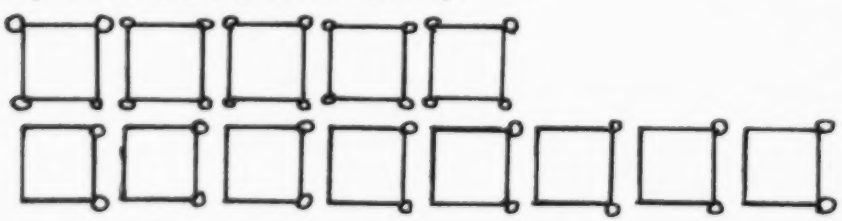
A student might construct the following table to solve the problem: "A recipe for pudding requires two cups of milk for every six servings. How much milk is needed for 24 servings?"

| CUPS OF MILK | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| NUMBER OF SERVINGS | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 |

## Sketching Diagrams and Pictures

Here is a possible sketch to help solve the problem:

> My friend owns a gas station. One day she sold four new tires for each of five cars and two tires for each of eight cars. How many tires did she sell that day?



## Relating the Problem to a Familiar One

In solving the following problem, a student might recognize the similarity with the previously solved carpenter shop problem:

In the crowded storeroom of a toy store bicycles and tricycles are so tightly packed that it is not easy to see how many there are of each. The manager of the store knows that there are at least two of each, and that there are twenty-three wheels. How many bicycles and tricycles are there?

## Recognizing Patterns

In solving the problem, "How many squares are on a checkerboard?" a student might count the number of 1 X 1 squares, 2 X 2 squares, 3 X 3 squares, and then recognize a pattern to determine the numbers of other squares:

| Size of Squares | Number of Squares |
|---|---|
| 1 X 1 | $64 = 8^2$ |
| 2 X 2 | $49 = 7^2$ |
| 3 X 3 | $36 = 6^2$ |
| 4 X 4 | $? = 5^2$ |
| 5 X 5 | = |
| 6 X 6 | = |
| 7 X 7 | = |
| 8 X 8 | $1 = 1^2$ |

## Manipulating Objects

The problem, "How many ways can you make change for a quarter?" could be solved physically using coins to show each possible way.

## Using Mental Arithmetic

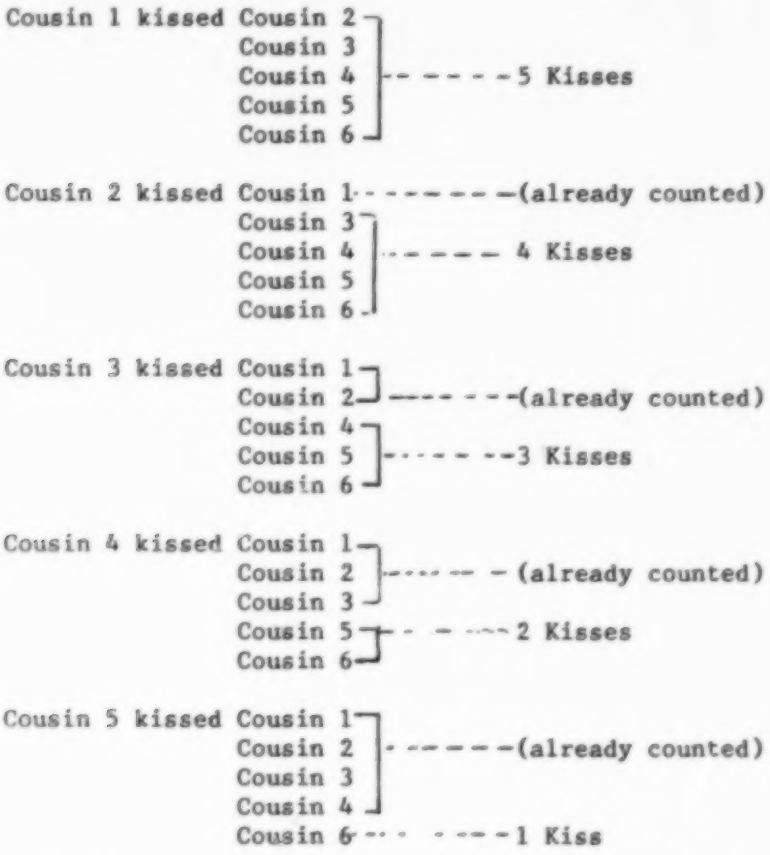
In the pennies problem the student who can divide and multiply two-digit numbers by 3 and 4 in his or her head will be more likely to reach a solution than the student who must use paper and a pencil. The focus can then be on the relationships in the problem rather than the computation.

## Recognizing the Reasonableness of an Answer

In solving the problem: "A box of number two pencils contains 25 pencils and costs 87¢. How much will 75 pencils cost?" a student who erroneously multiplies 75 X 87¢ could recognize that $65.25 is much too high to pay for 75 pencils. So, the student will reconsider the problem situation and be more able to correct the solution process.

## Making Organized Lists

Consider the problem: "Six cousins met at a family reunion. Each cousin kissed each of the other cousins just once. How many kisses were given?" The following organized list could be useful:




All cousin 6's kisses were counted already. The answer is 15.

## Breaking the Problem Down into Subproblems

Consider the problem, "How many squares are in a checkerboard?" Instead of coping with the entire 8 X 8 board at first, a student could think about how many squares there would be on a 1 X 1 grid, a 2 X 2 grid, a 3 X 3 grid, and so on. Then, to consider how many squares are on a 3 X 3 grid for example, he or she might break that problem down into how many 1 X 1 squares, 2 X 2 squares, 3 X 3 squares.

## Looking at a Problem in Another Way

In the toothpick puzzle, "Use 6 toothpicks to make 4 equilateral triangles," students need to break the mind set which leads them to think about the solution to the problem only in two dimensions.

## Generalizing

Often students can benefit from substituting smaller numbers or simple combinations in a problem to figure out the solution method and then generalize the method to the more complex calculations. For example, if a problem requires the computation of the number that 15 is 75% of, substituting 10% at first can help a student determine a solution method which can be transferred to the actual computation.

## Representing Information Algebraically

The ability to represent the problem data algebraically may be a useful skill in exploring the relationships in the following problem:

> A farmer buys 100 animals for $1000. How many of each animal does she buy if hens cost $7.50 each, pigs cost $25.00 each, and goats cost $30.00 each?
>
> H represents the number of hens, P represents the number of pigs, and G represents the number of goats;
>
> $$H + P + G = 100$$
> and $$7.50\ (H) + 25\ (P) + 30\ (G) = 1000$$

THE PROBLEM WITH PROBLEM SOLVING

Activity I.5

Form small groups of three to five. Ask the participants to discuss the implications of the results of the Second National Assessment of Educational Progress (NAEP) in Mathematics which are listed in the workshop packet. You may want to read some reference articles that contain analyses of the NAEP in order to prepare yourself for this activity (Amick, 1981; Carpenter, et al., 1980; 1981; 1983; Fennema, et al., 1981). Each group should choose a recorder to be prepared to report back to the entire group. Allow 10 to 15 minutes for the small-group discussion. Then spend five to ten minutes sharing the findings among the groups. As you take notes on a blank overhead transparency, list general implications of the results on the left-hand side and any possible instructional implications on the right.

Some of the general observations that could be made are listed below:

- One-step problems with addition and subtraction of whole numbers do not cause great difficulty; one-step problems with multiplication and division of whole numbers, and computation of rational numbers are somewhat more difficult.

- Students did almost as well on pure computation exercises with addition of fractions as on one-step word problems requiring comparable calculation.

- Extraneous information in a problem causes a reduction in the number of students who can solve it.

- Multi-step and non-routine problems caused great difficulty.

- Students perceive the study of mathematics as the memorization of a collection of facts.

- Rather than attempting to reason out the solution to a problem, many students merely tried to apply a single arithmetic operation to some of the numbers in the problem. Overhead Transparency G humorously illustrates this point.

- Incomplete understanding of computational algorithms (e.g., division, fraction operations) may lead to difficulties in problem solving.

- Reading difficulty cannot account for the poor problem-solving performance of students.

After analyzing the results, lead the discussion to address possible instructional implications from these results. Some instructional implications may have already been suggested by the groups.

Some instructional implications from this and other research on problem-solving are listed below:

- Devote more time to problem solving.

- Introduce more multi-step, non-routine problems.

- Present more problems containing unnecessary or insufficient data.

- Teach general and specific problem-solving strategies.

- Teach estimation and approximation skills.

- Use calculators and microcomputers to expand the repertoire of problem-solving strategies.

- Model good problem-solving behavior by engaging in problem-solving discussions in the classroom and by conveying respect for each student as a problem solver.

- Encourage the use of diagrams as aids in solving problems.

- Teach students to read and talk mathematics.

The following information on how the NAEP was conducted may be useful to you in leading the discussion:

- The second NAEP in mathematics was completed during the 1977-78 school year.

- The representative sample consisted of 70,000 9-, 13-, and 17-year-olds.

- Five basic content areas were assessed: number, variables and relations (algebra), geometry, measurement, and other topics (probability and statistics). The area of number included items on whole number concepts and computation, fractions and decimals, percents, integers, and word problems.

- The goals of the assessment are to make comprehensive data available on specific educational attainment of students, and to measure change in their attainments.

- The word problems were presented orally by a tape recording along with the written problems.

## DIAGNOSING STUDENT DIFFICULTIES IN MATHEMATICAL PROBLEM SOLVING

### Activity I.6

Use Overhead Transparency H to introduce a problem from the High School Proficiency Test. Lead a brainstorming session with the participants to generate a list of possible sources of student error in this problem. Record all suggestions made on a blank transparency or chart paper. You may need to probe to clarify meaning, but you should not be evaluating the suggestions. Some of the possible causes of student difficulty identified by the group might be:

- The number word is used rather than the numeral, "3";

- Extraneous data are included in the problem;

- Computation with time before and after the noon hour is required;

- Multiple steps are required; and

- Conversion between minutes and hours is necessary.

After you have gathered the information, lead a discussion on the sources of errors. Solicit from the group some categories of error, such as errors involving computation, interpretation of language, recording or organization, that are suggested from the specific items listed.

### Activity I.7

Divide the participants into small groups. Allow 30-45 minutes for the groups to work. If there are time constraints, you may choose to assign different work samples to each group or to allow each group to select a subset of the samples to analyze.

The task for each group is to analyze the work samples in terms of the problem-solving skills exhibited and the difficulties encountered by the students. You may do the analysis of one of the samples with the whole group to illustrate the kinds of observations that can be made.

Debrief the activity for about 10 minutes. Ask each group to share one observation. If time permits, you can entertain other comments from each group. It is important to allow time for each group to share something they discussed.

Some general points that you may want to weave into the discussion are:

- Different problems bring out different problem-solving behaviors of students;
- You learn more about problem-solving behavior from oral interviews than you do from an analysis of written work samples, and more from written work samples than you do from just the incorrect answer;
- Accurate diagnosis requires an analysis of a combination of oral and written evidence;
- Students may be using excellent problem-solving strategies and nevertheless get wrong answers; and
- Some students who get right answers are poor problem solvers and poor computers.

Assign the follow-up activity to be completed between this first workshop session and the next. Stress your interest in hearing feedback at the next session about the findings. You may want to collect the feedback forms. If any of the participants wants to tape the interviews, the group could listen to excerpts in the next workshop.

## CONCLUSION OF THE WORKSHOP

Summarize the contents of the workshop in relationship to the objectives (Overhead Transparency A). In general, you want to use this summary to set the context for future workshops. This workshop provides an overview of the problem-solving process--the definition of a problem and problem solving, and the introduction of problem-solving skills. In addition, the session serves as an introduction to planning instruction to meet student needs--diagnosing student difficulties in problem solving, examining the processes they use, and generating instructional suggestions.

Project Overhead Transparancy I to structure a description of the future sessions. Encourage participants to share feedback and successful problem-solving activities during the course of the training.

Training Materials for Module I

## OBJECTIVES FOR WORKSHOP I

THE OBJECTIVES OF THIS WORKSHOP ARE:

1. TO EXAMINE THE NATURE OF A PROBLEM AND THE PROBLEM-SOLVING PROCESS;
2. TO EXPLORE THE SKILLS NEEDED FOR EFFECTIVE MATHEMATICAL PROBLEM SOLVING; AND
3. TO DEVELOP THE ABILITY TO ANALYZE STUDENT DIFFICULTIES IN MATHEMATICAL PROBLEM SOLVING.

# PUZZLES

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WEATHER
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Overhead Transparency C

# PILES OF COUNTERS

CAN 23 COUNTERS BE PUT IN PILES SO THAT EACH PILE HAS THREE OR FOUR COUNTERS? PROVE YOUR ANSWER.

## PENNIES PROBLEM

BARBARA HAS SOME PENNIES. SHE HAS EXACTLY 3 TIMES AS MANY AS AL AND EXACTLY 4 TIMES AS MANY AS CARLOS. THEY HAVE LESS THAN 100 PENNIES ALTOGETHER. WHAT CAN YOU SAY THAT ISN'T OBVIOUS? HOW MANY PENNIES MIGHT EACH CHILD HAVE?

Overhead Transparency E

## DEFINITION

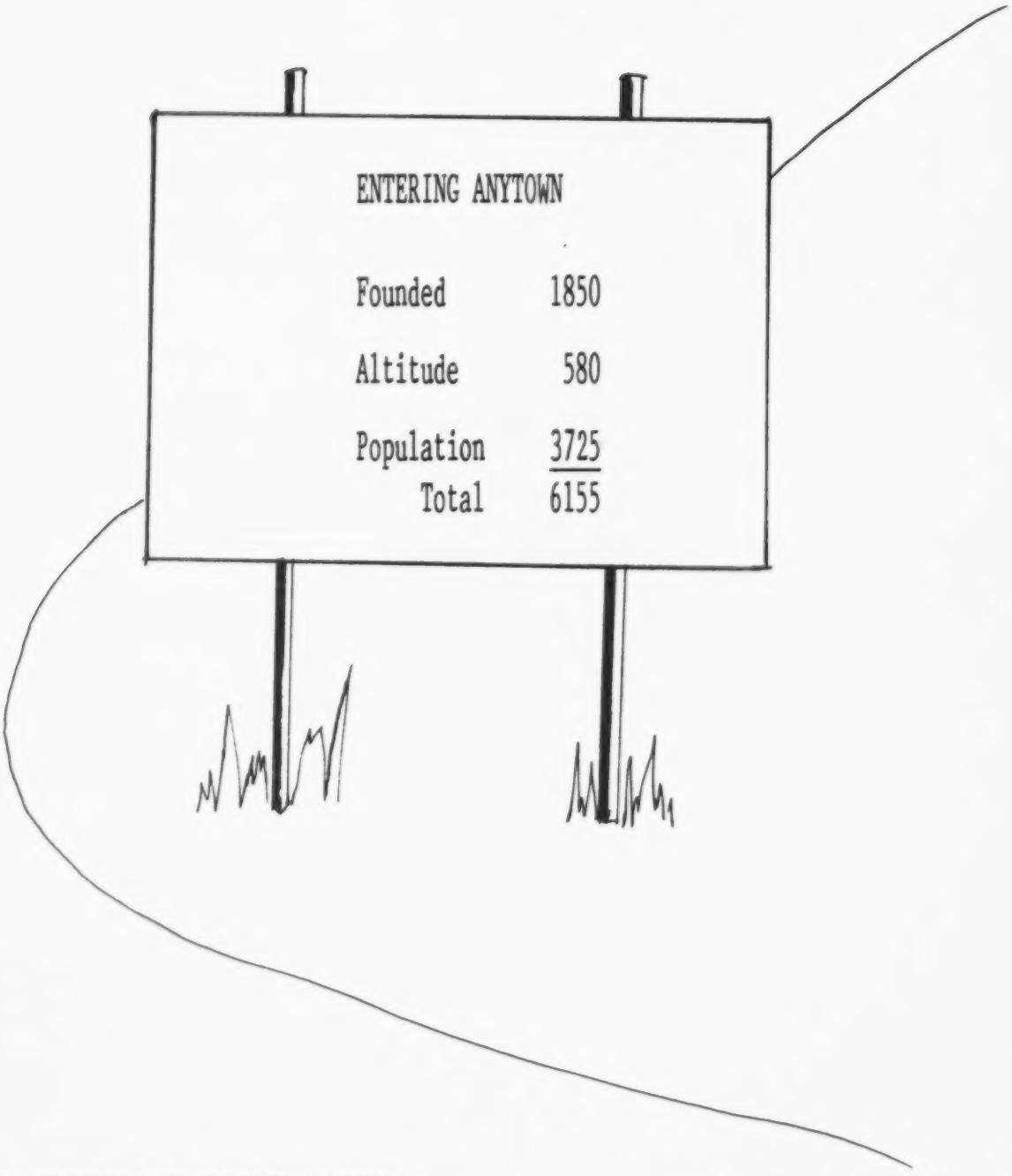
PROBLEM SOLVING IS THE PROCESS OF APPLYING PREVIOUSLY ACQUIRED KNOWLEDGE TO NEW AND UNFAMILIAR SITUATIONS.

Overhead Transparency F

## CARPENTER SHOP PROBLEM

IN MY CARPENTER SHOP I MAKE THREE-LEGGED STOOLS AND FOUR-LEGGED TABLES. ONE DAY I LOOKED AT MY DAY'S OUTPUT AND COUNTED 31 LEGS. HOW MANY STOOLS AND HOW MANY TABLES MIGHT I HAVE MADE?

Overhead Transparency G




THE PROBLEM WITH PROBLEM SOLVING

## PROBLEM FROM THE 1984 HIGH SCHOOL PROFICIENCY TEST

THE JOHNSON FAMILY MADE A 250 - MILE TRIP IN THEIR CAR. THEY LEFT HOME AT 8:00 A.M. AND ARRIVED AT THEIR DESTINATION AT 2:40 P.M. THE SAME DAY. IF THEY TOOK THREE 15 - MINUTE BREAKS TO REST AND A 50 - MINUTE BREAK FOR LUNCH, HOW MUCH TIME DID THEY ACTUALLY SPEND DRIVING?

A. 1 hour, 35 minutes

B. 5 hours, 5 minutes

C. 5 hours, 45 minutes

D. 6 hours, 40 minutes

Overhead Transparency I

## TOPICS

WORKSHOP 1. PROBLEM SOLVING: THE MOST BASIC MATHEMATICS SKILL

WORKSHOP 2. FOCUS ON EFFECTIVE PROBLEM-SOLVING STRATEGIES

WORKSHOP 3. EXPANDING WORD PROBLEM-SOLVING EXPERIENCES

WORKSHOP 4. PROBLEM SOLVING USING MODELS AND MANIPULATIVES

## GENERAL CONTENTS OF EACH SESSION

REVIEW OF RELEVANT RESEARCH

PROBLEM-SOLVING ACTIVITIES

SMALL AND LARGE GROUP DISCUSSION

PLANS FOR CLASSROOM IMPLEMENTATION

RESOURCES

# MODULE II

## FOCUS ON EFFECTIVE PROBLEM-SOLVING STRATEGIES

PACKET FOR WORKSHOP II

Focus on Effective Problem-solving Strategies

New Jersey State Department of Education
Division of Compensatory/Bilingual Education
Spring, 1985

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## INTRODUCTION TO WORKSHOP II

General and specific problem-solving strategies can be taught. General problem-solving strategies provide an overall procedure for attacking a problem. This framework can give students a sense of security. Problem solving creates tension; a general strategy can help make that tension productive. In this workshop Polya's four-step model for approaching the problem-solving process will be proposed as a general strategy.

Knowledge of a wide variety of specific strategies enables students to solve more problems. These specific problem-solving strategies can be applied within a general strategy. Thus, this workshop will address techniques for teaching specific strategies, and suggest problems that are appropriate to particular strategies.

The key to teaching problem-solving strategies is to ask students the types of questions they need to learn to ask themselves, that is, to model desired questioning behavior. The teacher should ask questions that are general enough to be applicable to most problems. For example, questions such as, "What is the unknown? and "What information is given?" help the student explore the meaning of any problem.

The other aspect of helping students learn problem-solving strategies is deciding when to intervene. If the teacher helps too much, the student need not think and loses even the illusion of working independently. The more dependent upon the teacher for problem solving, the less the student experiences the satisfaction of discovering the solution that motivates further problem solving. However, leaving the student alone too long with a problem-solving difficulty keeps the student from making progress. The best assistance from a teacher is unobtrusive and well-timed. The questions asked to help a student over a hurdle should lead the student to follow his or her own reasoning, to carry out the problem-solving process he or she had begun.

## OBJECTIVES FOR WORKSHOP II

The objectives for this workshop are:

1. To improve instructional techniques for teaching effective problem-solving strategies and skills; and

2. To identify appropriate problems for teaching specific problem-solving strategies.

## FOLLOW-UP: STUDENTS' PROBLEM-SOLVING BEHAVIORS

### Feedback Activity

Share your experiences in doing the follow-up activity from the last workshop. In particular, discuss the problem-solving strategies and skills used by your students. Record strategies and skills that other participants identify during the discussion.

## A TAXONOMY OF STRATEGIES AND SKILLS

### Activity II.1

It is useful to look at specific problem-solving strategies and skills within a framework. The taxonomy used here was proposed by Polya (1945) and has been adopted and adapted for use in many problem-solving programs. For example, the Iowa Problem Solving Project used the four-step model to develop an alternative approach to problem-solving evaluation (Schoen, Oehmke, 1980). The four steps are: understanding the problem, devising a plan, carrying out the plan, and looking back.

Using Polya's model, categorize and consolidate your lists of problem-solving strategies and skills from the follow-up activity and from Activity I.3 in Workshop I. This taxonomy is a general strategy for attacking problems.

Step 1: Understanding the Problem

Step 2: Devising the Plan

Step 3: Carrying Out the Plan

Step 4: Looking Back

## TEACHING A GENERAL STRATEGY FOR PROBLEM SOLVING

Activity II.2

Use Polya's steps to structure your solution to the following problem. Think about questions you might ask students to help focus their problem-solving processes.

### How many squares are on a checkerboard?

Step 1: Understanding the Problem

Step 2: Devising a Plan

Step 3: Carrying Out the Plan:

Step 4: Looking Back

Activity II.3

Solve one of the following problems in groups of six. One member of the group will be assigned the role of the teacher, one person will be the observer, and the other four will be the problem solvers. Follow Polya's four-step, general approach. Be aware of the specific problem-solving strategies that are used as well.

Problem A: The list of words that follow can be used to make impressive-sounding, three-word phrases. By choosing a word at random from each column in order, you can create pedantic phrases.

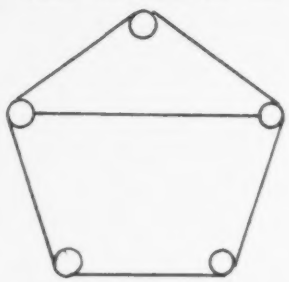
| 1 | 2 | 3 |
|---|---|---|
| functional | delivery | concept |
| systematic | transitional | process |
| consolidated | management | idea |
| coordinated | organizational | plan |
| integrated | implementation | program |

"Functional management process" and "integrated implementation concept" are two possible phrases. How many different three-word phrases can be made from these lists?

Problem B: A job pays 1 cent the first day, 2 cents the second day, 4 cents the third day, and so on, doubling the amount of payment each day. On what day would the pay be at least one million dollars?

Problem C: At a party every person shakes hands with every other person once. How many people were at the party if 15 handshakes took place?

Problem D: Put each of the numbers 1, 2, 3, 4 and 5 in one of the circles in this figure so that no number is connected directly to a consecutive number.



Problem E: In the crowded storeroom of a toy store bicycles and tricycles are so tightly packed that it is not easy to see how many there are of each. The manager of the store knows that there are at least two of each, and that there are twenty-three wheels. How many bicycles and tricycles are there?

Problem F: Camille had a bag of apples. She met her friend Anna and gave her half the apples plus half an apple. Later, she met Alvin and gave him half of her remaining apples plus half an apple. Then, she met Douglas and gave him half her remaining apples plus half an apple. Camille's bag was empty now. How many apples did she have in the beginning?

PROBLEM-SOLVING NOTES

Step 1: Understanding the Problem

Step 2: Devising a Plan

Step 3: Carrying Out the Plan:

Step 4: Looking Back

Activity II.4

Trial and error is a specific problem-solving strategy that is effective in tackling the following problems. Follow Polya's four-step, general strategy and apply trial and error to solve the problems.

Use the worksheet to keep track of specific problem-solving strategies that you use as you solve the problems.

Problem A: Make $1.00 using exactly 50 coins.

Problem B: A farmer buys 100 animals for $1,000. How many of each animal does she buy if hens costs $7.50 each, pigs cost $25.00 each, and goats cost $30.00 each?

Problem C: Decode this addition exercise. Each letter represents a digit:

```
 Send
+More
Money
```

After you have solved one or more of these problems, discuss the strengths and weaknesses of trial and error as a problem-solving strategy.

PROBLEM-SOLVING NOTES

**Step 1: Understanding the Problem**

**Step 2: Devising a Plan**

Step 3: Carrying Out the Plan:

Step 4: Looking Back

Activity II.5

Drawing diagrams or pictures is a powerful problem-solving tool. A picture can help you explore the meaning of a problem, point the way to a solution, and evaluate whether your solution works.

Draw a picture or diagram to help solve these problems. Use the worksheet to take notes.

Problem A: On Tuesday, the temperature dropped from 11°F in the morning to 3°F at night. By Wednesday morning it was - 8°F. How much had the temperature dropped since Tuesday morning?
(From the New Jersey High School Proficiency Test administered in Spring, 1984).

Problem B: Five children are running a race. Halfway through the race, they are in these positions:

Leroy is 20 meters behind Rachel.
Rachel is 50 meters ahead of Derrick.
Derrick is 10 meters behind Carla.
Steve is 30 meters ahead of Leroy.
Carla is 50 meters behind Steve.

At this point, who is winning the race? Who is second? third? fourth? fifth? Did you need to have all of the above information to solve the problem?

Problem C: Hiroko is bigger than Suzanne. Suzanne is bigger than Jennie. Who is the smallest?

Problem D: Sybil is less twerty than Charisse. Percy is twertier than Charisse. Who is twertiest? Who is least twerty?

Problem E: Each day a baker makes a large amount of dough for bread. He divides the dough into six equal pieces and rolls out each piece on a long board. Then he cuts the piece he has rolled into four equal parts. Each of these parts is baked in the oven to produce a loaf of bread which is 1 1/2 feet long. He does the same thing with each piece of dough he started with. If all the loaves of bread that he made in one day were lined from end to end, how long would the bread line be?

Problem F: My friend owns a gas station. One day she sold four new tires for each of five cars and two tires for each of eight cars. How many tires did she sell that day?

Problem G: Six cousins met at the family reunion. Each cousin kissed each of the other cousins just once. How many kisses were given?

Problem H: Tanya saved boxes. She packed them inside each other so they would take up less room in her closet. She had four big boxes. Inside each of the big boxes there were two middle-sized boxes. Inside each middle-sized box were five tiny boxes. How many boxes did she have altogether?

Evaluate the pictures and diagrams drawn to solve the problems. How could each be improved to more efficiently and simply convey the relationships expressed in the problem?

PROBLEM-SOLVING NOTES

Step 1

Step 2

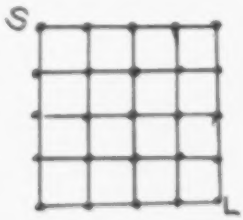
Step 3

Step 4

Activity II.6

Transforming a problem into a simpler problem or breaking it down into subproblems can help make a complex problem manageable. Practice this strategy with the following problems:

Problem A:

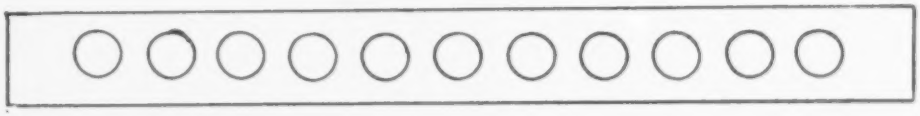



This is a map of part of the city where Andrea lives. Her house is shown by the dot labeled "S." Andrea often walks to the library. The library is located at "L." She always follows the streets (no diagonals) and always takes the shortest routes. She likes to take different routes each time she goes to the library. How many different shortest routes can she take to the library?

Problem B: If you fold a paper 1mm thick, 10 times, how thick will the paper be?

Problem C: How many different shapes can you cut from grid (graph) paper using just two squares of the grid in each shape? three squares? five squares? eight squares?

Problem D: Construct a peg puzzle from a cardboard shoe box cover and ten golf tees (five in one color, and five in another). Use one of the golf tees to punch 11 evenly spaced holes lengthwise in the middle of the shoe box cover as shown below.

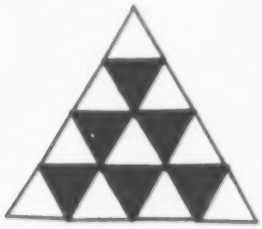


Place five pegs of the same color in the first five holes on the left side, and five of the other color in the last five holes on the right side. The object of the game is to interchange each group of pegs with the group of the opposite color. The rules for moving pegs are:

1) You can move a peg to the adjacent empty hole;
2) You can jump over one peg only if it is of the opposite color; and
3) The pegs on the left move only to the right and those on the right move only to the left.

What is the fewest number of moves required to move all the pegs of one color to the other side? [Hint: How many moves would it take if you have one peg of each color and used only three of the holes?]

Problem E: How many triangles are there in this figure?



Problem F: You have 2 unmarked jugs, one that holds 3 liters and one that holds 4 liters. Using only the 2 jugs, how can you measure exactly 2 liters of water?

PROBLEM-SOLVING NOTES

Step 1

Step 2

Step 3

Step 4

Activity II.7

Constructing a chart or table is a useful strategy to systematically organize a solution to some problems. Practice making charts and tables with the following problems:

Problem A: How many different ways can you make change for a quarter?

Problem B: A recipe for pudding requires two cups of milk for every six servings. How much milk is needed for 24 servings?
[From the N.J. High School Proficiency Test administered in Spring, 1984]

Problem C: Carmen, Tony, and Kim live in adjacent apartments. Tony lives in the middle apartment. They work as an electrician, a computer operator, and an automobile mechanic, but not necessarily in that order. The computer operator walks Kim's dog when she is not home. The electrician lives next to Carmen. What occupation does each person have?

Evaluate the charts and tables you made. How could they be improved? What other problems can you think of that could be efficiently solved by this strategy?

PROBLEM-SOLVING NOTES

Step 1

Step 2

Step 3

Step 4

NAME _____

DATE _____

## FOLLOW-UP ACTIVITY FOR WORKSHOP II

Plan and teach a lesson to your class on the general four-step strategy for problem solving or one of the specific strategies. Choose some appropriate word problems from your textbook. You can use problems from this workshop.

After you teach the lesson, record your reactions and comments below.

_____

_____

_____

_____

_____

_____

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_____

_____

_____

_____

_____

_____

_____

_____

Was your textbook a good source of problems for the purpose of this activity?

Bring a copy of your textbook(s) to the next workshop.

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WORKSHOP PRESENTER'S GUIDE

Module II: Focus on Effective Problem-solving Strategies

## Trainer Overview of Module II

WORKSHOP OBJECTIVES:

1. To improve instructional techniques for teaching effective problem-solving strategies and skills; and

2. To identify appropriate problems for teaching specific problem-solving strategies.

TRAINING MATERIALS:

Workshop Packet II

Overhead Transparency A: Objectives for Workshop II
Overhead Transparency B: General Clues from Research on Teaching Problem-solving Strategies
Overhead Transparency C: Polya's Taxonomy
Overhead Transparency D: Checkerboard Problem
Handouts for Activity II.3: Directions
Blank Transparencies
Chart Paper and Markers
Hand-held Calculators
Grid Paper

PREPARATION SUGGESTIONS:

- Listen to and analyze in advance any tapes of student problem-solving behavior recorded by the participants.

- Read the procedures and anticipate questions that may be asked by the participants.

- Practice using each strategy to solve all the problems. Develop a sense of which strategies are effective with which problems.

- Arrange the list of specific problem-solving strategies and skills generated in Workshop I according to Polya's four-step taxonomy. Prepare an overhead transparency or chart of the list.

- Prepare the overhead transparencies, copies of the direction cards for Activity II.3 (one set for each five participants), and sufficient copies of the workshop packets.

- Write the problems from Activity II.3 on pieces of chart paper so that each group can be given two problems.

## Procedures for Workshop II

### INTRODUCTION

Use Overhead Transparency A to introduce the objectives for the workshop.

### FOLLOW-UP: STUDENTS' PROBLEM-SOLVING BEHAVIORS

Ask the participants to share their experiences in doing the follow-up activity. Use a blank overhead transparency to take notes on the responses. Play excerpts of any tape recordings that were made. Keep track of the problem-solving strategies and skills that students used.

### A TAXONOMY OF STRATEGIES AND SKILLS

#### Activity II.1

Use Overhead Transparency B to structure a discussion of teaching students to use problem-solving strategies. You may want to read some reviews of research to prepare yourself to lead the discussion (Driscoll, 1982; Suydam, 1980). You want to stress that problem-solving strategies and skills can be taught and practiced. Some information that may be useful in leading the discussion follows:

- The terms "strategy" and "heuristic" are synonymous. A heuristic or strategy is a problem-solving method that is general enough to be applicable to a variety of problems and subjects, yet specific enough to help the problem solver approach, understand, and/or efficiently organize a solution.

- Problem solving skills include the ability to recognize superfluous data, identify missing data, use mental arithmetic, use computational algorithms, construct and interpret a graph or table, and estimate. Many of these skills appear as separate topics in the mathematics curriculum. The application of these skills in problem solving should be integrated into instruction. Some of these skills--identifying missing data and recognizing superfluous data, for example--will be included in the next workshop.

- The distinction between a strategy and a skill is often cloudy. In this workshop one of the activities focuses on drawing diagrams or pictures for problem solving. The ability to draw an effective diagram can be viewed as a skill. The use of that skill to approach particular problems is a strategy. It is not necessary for the purposes of this training to draw this distinction between strategies and skills. However, you as the trainer may need to think about it to prepare for questions.

Use Overhead Transparency C to introduce a general strategy, Polya's four-step taxonomy of problem-solving strategies and skills. Elicit examples of skills and strategies which fall into each category to clarify the taxonomy. Give each participant five minutes to consolidate the lists of problem-solving skills and strategies from the follow-up activity and from the third activity in Workshop I using the taxonomy. Then, present your consolidated list and ask for additions from the participants. Note that the same strategies and skills may appear in more than one category, since the same strategy or skill may be useful in more than one phase of the problem-solving process. For example, trial and error can be an effective strategy to explore the meaning of a problem, or to reach a solution.

## TEACHING A GENERAL STRATEGY FOR PROBLEM SOLVING

### Activity II.2

Lead the group in solving the checkerboard problem (Overhead Transparency D) using Polya's steps to structure the questions you ask and the suggestions you make. Think carefully about your teaching behavior, since you want to model good questioning techniques. Choose questions or suggestions that don't give answers, but that help the problem solvers focus their thinking. Remember that solving the problem is one goal, but that teachers want to encourage students to ask these questions for themselves in solving future problems.

Even though the four-stage model appears sequential, stress to the group that it should not be interpreted as a fixed, step-by-step formula to get a solution. For many problems, you need to move back and forth among the stages. Once you understand some problems, you may move immediately to solve them. The four-stage model is designed to help guide students in attacking problems; it should not be used to trap students in a rigid approach to the problem-solving process.

Some possible questions in each category of Polya's model are listed below. The lists were not designed to be used as a script for problem solving. You may want to use some of them to begin your interaction with the problem solvers, but you will want to respond to the suggestions made by the group in choosing your next question or suggestion. Each of the strategies and skills identified by the group in the last activity could be transformed into a general question or suggestion to stimulate the problem-solving process.

#### Step 1: Understanding the Problem

What are you trying to find out?
What is the unknown quantity or information?
What information do you have?
What do you know about the situation?
What information is assumed?

What further information do you need?
What is the relationship between the various pieces of information in the problem?
What questions do you have about the situation?
Draw a picture to help you understand the problem.
Try a specific case to help you learn about the problem.
Is it possible to solve the problem?
Is this a reasonable problem? Why or why not?
Outline the information given and implied in the problem.
Restate the problem in your own words.

Step 2: Devising a Plan

Do you know a related problem?
Think of a familiar problem where you have to answer a similar question.
Here is another problem related to yours. Could you use it to help you?
Try to solve a simpler problem.
Drop a condition and then reimpose it.
Deal with specific examples or cases.
Can you develop a pattern?
Guess a solution or first step.
Can you write an equation?
Display examples or cases in a chart or table.
Make a hypothesis; generalize from some cases or examples.

Step 3: Carrying Out the Plan

Can you extend the pattern you recognized?
Apply your solution method for a simpler problem to the more complex one.
What computational algorithms could you use?
Check each step.
Prove that each step is correct.
Describe your solution process as you carry it out.

Step 4: Looking Back

Check to make sure that the solution satisfies the conditions.
Did you use all the relevant data?
Describe your solution process.
Could you have solved the problem another way?
Can you use the same method for another problem?
What problems are similar to this one?
Is the answer reasonable? How do you know?
Compare your solution to your guess or estimate.
What are the advantages and disadvantages of your solution method?
What were the pitfalls in this problem?
Verbalize the stumbling blocks you overcame to solve the problem.
Which solution strategies were most helpful in solving the problem?

Debrief the problem-solving activity with the group. Ask the participants to identify the questions or suggestions made by you or other group members, or made to themselves, that were most effective. The next activity is designed to help them practice good instructional behavior.

Activity II.3

Divide the participants in groups of five. Give each group a set of directions to be distributed. One group member will be assigned the role of teacher, one, of observer, and three, of problem solver. If a group has fewer members, there will be fewer problem solvers. Each group should have chart paper or a chalkboard for the teacher to use.

Although each member of the group knows the role of the others (teacher, observer, or problem solver), the specific directions given to each should not be shared until the end of the activity.

Assign two of the problems to each group by distributing two pieces of chart paper with a problem written at the top of each. Hand-held calculators should be available, particularly in Problem B. After the first problem is solved, the observer and the teacher can switch roles for the second problem. The problems are to be solved orally as a group. The problem solvers can make notes on the sheet in the packets.

After the groups have completed both problems, allow about ten minutes for the group members to share their observations among themselves. They should relate the behaviors observed to the assigned roles. Then bring the participants back into the large group, and ask the observers to share the highlights of their discussions.

## LEARNING SPECIFIC STRATEGIES

Activities II.4-II.7

Each of these activities focuses on one specific strategy. You may choose to cover as many of them as time allows. A minimum of 30 minutes will be needed for each strategy. The procedure for each is essentially the same, so they are described together.

For each activity, introduce the strategy by modeling how you would use it to solve a problem. Think aloud as you engage in the problem-solving process. Try to make some mistakes; you want to illustrate that when you listen to your own thoughts, you find many of your own errors. You may use one of the given problems in the activity, a previously posed problem, or one of your own choosing. Remember to use the specific strategy in the context of Polya's general strategy.

Allow about 15-20 minutes for participants, working alone or in pairs, to practice using the strategies with the given problems. Then spend about 5-10 minutes discussing the strengths and weaknesses of the strategy in relation to the problems. You will want to analyze some of the solution strategies with the group. For example, you might have someone draw a picture or diagram constructed in Activity II.5 on a blank overhead transparency. Then, the group can make suggestions or modifications. Here you are leading the group to carry out the fourth step of Polya's general strategy "Looking Back." Participants may wish to share other problems that are appropriate to the strategy.

Some of the problems require special materials. These requirements are listed below to help you prepare for the activity:

Activity II.6

Problem B - Hand-held calculators
Problem C - Grid (graph) paper and scissors
Problem D - Shoe box covers and sets of ten golf tees (5 in each of two colors)

CONCLUSION

Summarize the session before assigning the feedback activity. Some of the points to make follow:

- If we expect students to use strategies in problem solving, we must teach a wide variety of strategies.

- Practicing a strategy is only the first step; students need to learn how to decide when to use each. This occurs when students have many opportunities to talk and write about their thinking processes, and to analyze the thinking of others.

- After instruction focusing on one specific strategy, pose problems that represent a range--some that could be appropriately solved by the strategy, and others that could not be--and help students compare and contrast the problems and solutions.

- Problem solving is learned through interaction--of the student and teacher, and between students--so engage your students in problem-solving activities in large and small groups, and in pairs, as well as individually.

- The teacher must think carefully about the roles he or she plays in helping students develop competency in the problem-solving process. Kantowski (1980) suggests the following teacher roles to correspond to the continuum of student problem-solving development:

  . The teacher serves as a model for students who have little or no understanding of problem solving;

- The teacher serves as a crutch and provides an atmosphere of trust to help students who are insecure about independent problem solving;

- The teacher becomes a problem provider for students who feel comfortable with problem solving; and

- The teacher serves as a facilitator for students who are able to search for and generate their own problems, select appropriate strategies, and evaluate the efficiency of various solutions.

In assigning the follow-up activity, you may want to provide planning time for the teachers during the workshop session. Teachers are asked in the assignment to choose some appropriate word problems from their textbooks. This exercise of examining textbook word problems is included to build a transition into the next workshop, "Expanding Word Problem-solving Experiences."

Training Materials for Module II

## OBJECTIVES FOR WORKSHOP II

The objectives for this workshop are:

1. To share feedback on the problem-solving behavior of the students of the participants;

2. To improve instructional techniques for teaching effective problem-solving strategies and skills; and

3. To identify appropriate problems for teaching specific problem-solving strategies.

## GENERAL CLUES FROM RESEARCH ON TEACHING PROBLEM-SOLVING STRATEGIES

- Give hints or suggestions rather than step-by-step procedures to follow.
- Give students less help than usual.
- Expose students to a wide variety of problems to improve their ability to flexibly approach problem-solving.
- Teach a variety of specific strategies or heuristics.
- Apply problem-solving strategies to the teaching and learning process in all areas of the curriculum.
- Use student errors to generate learning experiences.

## POLYA'S TAXONOMY

Understanding the Problem

Devising a Plan

Carrying Out the Plan

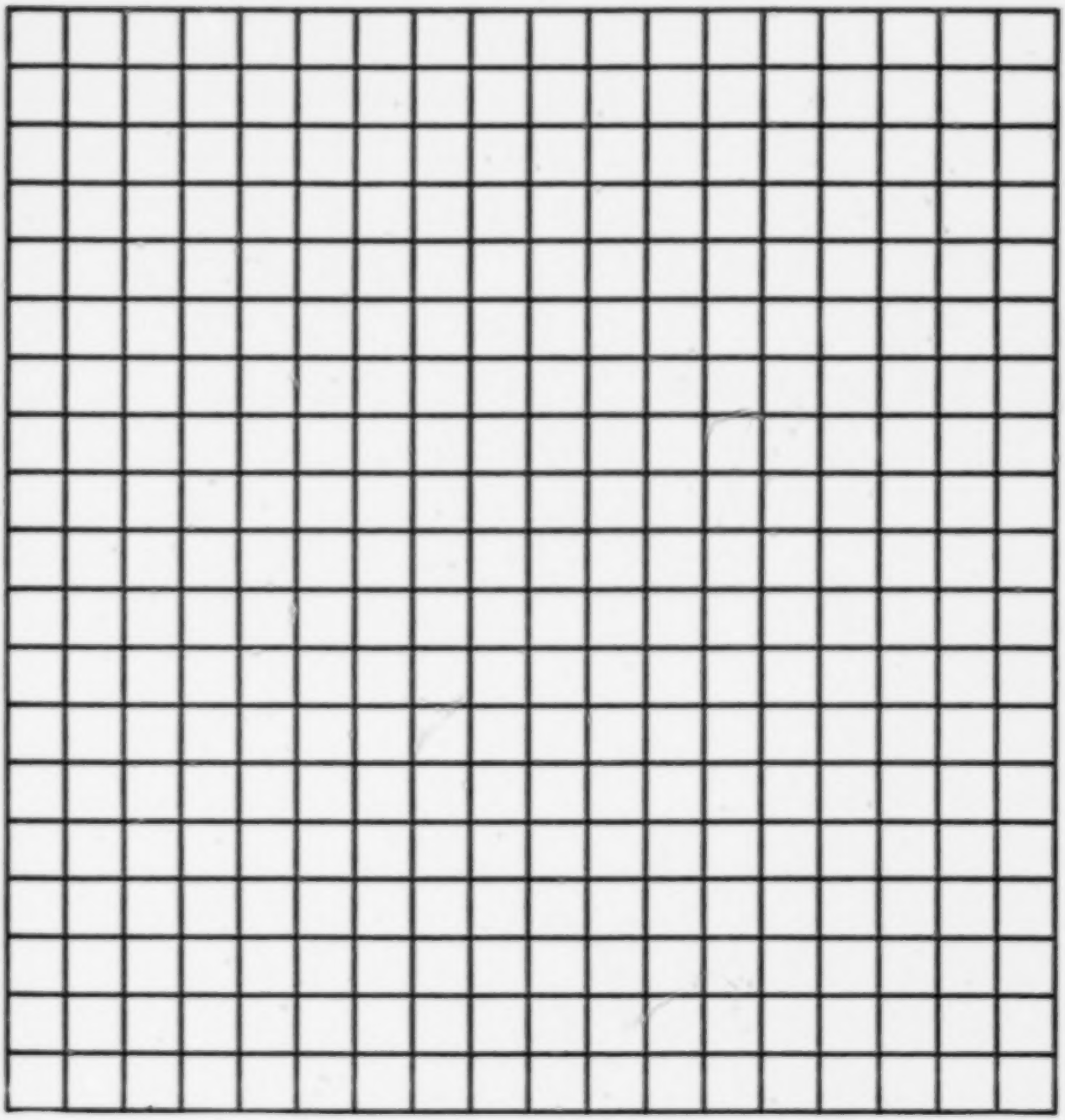
Looking Back

Overhead Transparency D

## CHECKERBOARD PROBLEM

How many squares are on a checkerboard?

GRID PAPER



YOU ARE TO TAKE THE ROLE OF THE TEACHER

Follow the attached directions.

YOU ARE TO TAKE THE ROLE OF OBSERVER.

Follow the attached directions.

DIRECTIONS FOR OBSERVER

Take notes on the verbal and non-verbal behaviors of the teacher and the problem solvers.

Write down the questions asked and suggestions given by the teacher.

What techniques did the teacher use to draw out the quiet problem solver?

What techniques did the teacher use to prevent the eager problem solver from telling everyone the solution?

How did the teacher respond to the questions of the problem solvers?

What other observations did you make of the problem solvers and the teacher?

YOU ARE PROBLEM SOLVER 1.

Follow the attached directions.

DIRECTIONS FOR PROBLEM SOLVER 1

Try to solve the problem. However, your role is to ask questions of the teacher. Ask specific how-to questions to see if you can get the teacher to tell you how to do the problem. You are trying to get the teacher to do your problem-solving work for you. [Of course, to the extent you are successful, you will be denied the opportunity to feel the thrill of meeting the challenge of problem solving.]

Problem-solving Notes

Step 1

Step 2

Step 3

Step 4

YOU ARE PROBLEM SOLVER 2.

Follow the attached directions.

DIRECTIONS FOR PROBLEM SOLVER 2

Try to solve the problem. However, your role is to be very eager to tell the teacher what the answer is and that you know how to do it, to monopolize the discussion. [To the extent that you are successful, you will be depriving the other problem solvers of the opportunity to experience the satisfaction of reaching solutions on their own.]

Problem-solving Notes

Step 1

Step 2

Step 3

Step 4

YOU ARE PROBLEM SOLVER 3.

Follow the attached directions.

DIRECTIONS FOR PROBLEM SOLVER 3.

Try to solve the problem. However, your role is to be very quiet. You wait for the teacher to draw you out. [To the extent you are successful, the teacher will know very little about your problem-solving behavior unless he or she makes an effort to probe.]

Problem-solving Notes

Step 1

Step 2

Step 3

Step 4

# MODULE III

# EXPANDING WORD PROBLEM-SOLVING EXPERIENCES

PACKET FOR WORKSHOP III

Expanding Word Problem-solving Experiences

New Jersey State Department of Education
Division of Compensatory/Bilingual Education
Spring, 1985

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## INTRODUCTION TO WORKSHOP III

Students learn to problem solve by solving a variety of problems. However, the word problems provided by most textbooks only represent a narrow range of problem types. Certainly students need to be able to solve the typical textbook word problem, but using the textbook as a primary source of problems limits instructional opportunities.

Research indicates that students have difficulty when a problem:

- o Requires more than one operation or step;
- o Contains complex sentence structure;
- o Differs in type from the problems that precede it;
- o Contains extraneous information;
- o Has the data presented out of order; and
- o Contains large numbers (Suydam, 1980).

Students need to be given many problems which exhibit these features, if they are ever to learn to handle them. Non-routine and complex problems encourage the use of more sophisticated problem-solving skills.

This workshop will suggest techniques for enhancing the word problem experiences provided by the standard textbook and for generating problem-solving experiences from other readily available resources.

## OBJECTIVES FOR WORKSHOP III

The objectives of this workshop are:

1. To explore techniques for extending the word problem experiences provided by standard textbooks; and

2. To increase the ability of participants to generate problem-solving experiences from readily available resources.

## FOLLOW-UP: TEACHING PROBLEM-SOLVING STRATEGIES

### Feedback Activity

Share your experiences in doing the follow-up activity from the last workshop. What difficulties did your students have? Describe how you responded to their difficulties. Comment on your search for appropriate textbook problems.

## A LOOK AT TEXTBOOK WORD PROBLEMS

### Activity III. 1

The following problems are typical textbook word problems. Use the rules in "A Student's Misguide to Problem Solving" to solve these problems. What's your score?

1. A bus tour visited the Grand Canyon. 27 people went down into the canyon. Each person carried 2 sandwiches. How many sandwiches did the people carry?

   _____

2. Charles had 12 checkers, but he lost some. He has only 9 checkers left. How many checkers did he lose?

   _____

3. 80 boxes of shoes arrived at the store. There were 5 cartons with the same number of boxes in each carton. How many boxes were in each carton?

   _____

4. The Town Council paid $.14 for every kilogram of litter collected during a "Clean Community Drive." How much money was paid for 425 kilograms of litter?

   _____

5. One fifth grade class made anti-litter posters for their school's anti-litter campaign. They spent $2.45 on paper, $1.45 on paints, and $3.25 on decorations. How much money did they spend in all?

   _____

6. A tool and die maker kept a record. A machine she helped to make produced 1,896 parts in 12 hours. How many parts did it average per hour?

_____

7. Rosa is saving to buy a new aquarium. This week she saved $1.43. In all she has saved $7.29. How much had she saved before this week?

_____

8. There are 42 students in the gym, 58 students in the library, and 47 students in the auditorium. How many students in all?

_____

9. Some students will display 48 pieces of pottery. They will display 8 pieces on each table. How many tables will they use?

_____

10. Archie made 6 bead bracelets. He used 9 beads for each bracelet. How many beads did he use?

_____

Work Area

Activity III. 2

Analyze the textbook you use as follows:

Turn to a chapter or section on computation (e.g., addition of fractions). Look at the word problems at the end of the chapter.

How many problems are included? _____

How many require only the application of the content in the chapter? _____

How many problems require only one computation or step? _____

Comment on the content of the problems.

_____

_____

_____

_____

_____

_____

_____

_____

Scan the text to locate non-routine, multi-step, and mixed practice problems. Are these problems labeled in a special section? What are the advantages and disadvantages of the placement of these problems?

_____

_____

_____

_____

_____

_____

_____

_____

Look in the table of contents for the chapter(s) that specifically address problem solving. Examine and comment on the nature, content, placement, and difficulty of the problems included. Are the sections organized by problem content, computational demands, or problem-solving strategy? What are the advantages and disadvantages of the various approaches?

## IMPROVING TEXTBOOK WORD PROBLEMS

Activity III. 3

The following pages contain suggestions and examples for altering textbook word problems to provide more problem-solving experiences for students. Review the suggestions and the samples. Use those ideas to plan a problem-solving activity for your students that uses the textbook as a starting point. You may want to design an activity that provides reinforcement for the lesson you planned and taught after Workshop II.

Review the activities you planned in pairs. Give each other constructive feedback to improve the activities. You will try out the activity with your students as a follow-up to this workshop.

## Change the question(s) asked.

EXAMPLE:

Manuel set his typewriter so he could type 70 letters or spaces on a line. How many lines will he use to type 2800 letters or spaces?

ALTERNATE QUESTIONS:

(1) He typed the following text so that no words were divided: "My Uncle Jack often visited us in the summer. He enjoyed the hustle and bustle in our house and in our city. Although he loved his mountain house, he sometimes became restless with the quiet beauty." How many spaces were left on the last typed line?

(2) If the average length of each word is five letters, about how many words would fit on 8 lines?

## Write guided solutions to some textbook problems.

EXAMPLE:

A living room is 16 ft. by 20 ft. The rug, also rectangular, measures 12 ft. by 15 ft. How many square feet of the floor remain uncovered by the rug?

GUIDED SOLUTION:

What does the problem ask you to find?

_____

_____

_____

Draw a picture of the room and the rug. Label the dimensions of the room and the rug. Shade the area of the floor that is not covered by the rug.

What formula do you know that might help you solve the problem? How will you use it? What do you need to do to find an answer?

Check each step of your computation.

Compare your answer to the picture. Does it satisfy the conditions? Is it reasonable?

## Use computational examples as headlines or equations for assigning students to write story problems.

EXAMPLE:

Write a problem to go with each of the following:

1) $2{,}856 \div 3 = \square$

2) $2\ (15 + 3) = 36$

3) $\square = 2/3 \times 1\ 1/8$

4) Add \$5.25 and \$1.90, and then subtract the sum from \$10.00.

SAMPLE STORY PROBLEM:

2) 2 (15 + 3) = 36

I have 15 pet rabbits and 3 cats that live in my backyard. One night I looked out of my bedroom window and saw the eyes of all my rabbits and cats. How many eyes did I see?

## Use the pictures, charts, and graphs in the textbook as resources for students to use in writing word problems.

EXAMPLE:

A textbook has a line graph showing the population changes in New York, Chicago, and Los Angeles from 1978-1984. Sample problems that might be written are:

1) If the population of New York changes as much in 1985 as it did in 1984, what will the population be in the beginning of 1986 to the nearest 100,000?

2) Look up the population changes for three other cities in an almanac and add the data to the line graph. What city experienced the greatest growth in a one-year period?

3) What is the overall population change shown in the graph?

## Instead of assigning a whole page of problems, ask the students to find the problem that has a particular answer.

## Use the textbook as an environment for a "problem hunt."

EXAMPLE:

Hunt for a problem in your textbook that meets each of the following conditions:

1) Find a problem with an answer between 2,000 and 2,225.

2) Find a problem that has an answer greater than the square of a prime factor of 56.

3) Find a problem that asks you to give the remainder from a division computation as the answer.

4) Find a problem that has an answer in square meters.

## Erase the numbers in the question and give the answers.

EXAMPLE:

Diana had _____ goldfish. She put the same number in each of _____ fish tanks. How many goldfish did she put in each tank? Answer: 12

## Add superfluous data to the problems.

EXAMPLE:

Howard answered 94% of the 50 test questions correctly. How many questions did he answer correctly?

CHANGED PROBLEM:

Howard had a history test each week. Last week he answered only 34% of the questions correctly. This week he improved his score. He answered 94% of the test questions correctly. This week there were 50 questions on the test, twice as many as there were on last week's test. How many questions did he answer correctly this week?

## Delete some data so that the problem is unsolvable. Have the students identify the data that are needed.

EXAMPLE:

Orlando bought milk for $1.29 and a bag of onions for $1.39. How much change did he get?

What do you need to know before you can solve this problem?

## Scramble the order of the word problems in the text.

## Create multi-step problems from one-step problems.

## Sequence problems to help students generalize from smaller to larger numbers, from simpler to more complex combinations, and from whole numbers to fractions and decimal computation.

EXAMPLES:

1) Iris is now 14 years old. How many years old will she be in 2 years? _____ in 3 years _____? in 5 years _____? in 10 years _____? in 37 years _____? Can you describe how to solve this sort of problem no matter what ages and years are given?

2) Curtis got 10% of the problems wrong on a test. If there were 20 problems on this test, how many did he get wrong? _____ Complete the chart below. Notice the solution process you use. Will your solution process work in any such problem?

| # of problems | 20 | 40 | 80 | 160 | 320 |
|---|---|---|---|---|---|
| # wrong | | | | | |

## Eliminate the questions and ask students to discuss the situation posed.

EXAMPLE:

What can you say about purchases that can be made under the terms of this sale?

| | | |
|---|---|---|
| SALE | PENCILS | 15¢ EACH |
| | PENS | 39¢ EACH |
| | NOTEBOOKS | 75¢ EACH |
| LIMIT: | 4 OF EACH TO A CUSTOMER | |

POSSIBLE OBSERVATIONS:

If I buy one pencil and one pen, I would spend 54¢.

If I buy one of each item, I would spend $1.29.

The most I could spend would be $5.16.

If I spent 93¢, I might have bought 2 pens and a pencil.

Could I spend $2.97?

## USING OTHER RESOURCES TO

## GENERATE PROBLEM-SOLVING EXPERIENCES

Activity III. 4

Maps, almanacs, magazines, catalogs, newspapers, advertising circulars, and telephone books are some resources which can be effectively used to generate problem-solving experiences.

Use a supermarket circular to do the following activities:

You have only $10.00 to spend. Try to buy as many items as you can. You may use an item only once. Good luck! Only items listed in the circular can be used.

| | ITEM | PRICE |
|---|---|---|
| 1 | | |
| 2 | | |
| 3 | | |
| 4 | | |
| 5 | | |
| 6 | | |
| 7 | | |
| 8 | | |
| 9 | | |
| 10 | | |
| 11 | | |
| 12 | | |
| 13 | | |
| 14 | | |
| 15 | | |
| | TOTAL COST | |

Select 10 items from the circular. The person who spends the most will be the winner. You may use an item only once.

| | ITEM | PRICE |
|---|---|---|
| 1 | | |
| 2 | | |
| 3 | | |
| 4 | | |
| 5 | | |
| 6 | | |
| 7 | | |
| 8 | | |
| 9 | | |
| 10 | | |
| | TOTAL COST | |

Find as many ways as you can to spend exactly $15.23 from the circular.

Activity III. 5

Examine a newspaper and brainstorm ideas for problem-solving activities using the newspaper as a resource.

Activity III. 6

Form small groups. Each group should choose and develop one idea from the list of ideas for activities with the newspaper. One member of each group will be responsible for rewriting the group's activity. The activities for each group will be shared with all the participants for classroom use.

NAME _____

DATE _____

## FOLLOW-UP ACTIVITY FOR WORKSHOP III

Use the activity you designed in Activity III.3 with some of your students. Record your reactions and comments below.

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WORKSHOP PRESENTER'S GUIDE

Module III: Expanding Word Problem-solving Experiences

## Trainer Overview of Module III

WORKSHOP OBJECTIVES:

1) To explore techniques for extending the word problem experiences provided by the standard textbook; and

2) To increase the ability of participants to generate problem-solving experiences from readily available resources.

TRAINING MATERIALS:

Workshop Packet III
Overhead Transparency A: Objectives for Workshop III
Overhead Transparency B: A Student's Misguide to Problem-Solving
Overhead Transparency C: Answers
Overhead Transparency D: Key Words
Blank Transparencies and Markers
Mathematics Textbooks Used by Participants
Supermarket Circulars
Newspapers
Hand-held Calculators

PREPARATION SUGGESTIONS:

- Read the procedures and anticipate questions that may be asked by the participants.

- Review the textbooks used by the participants in terms of the questions asked in Activity III. 2.

- Review the list of suggestions for transforming textbook word problems and try out some of the activities with students.

- Prepare the overhead transparencies and sufficient copies of the workshop packets.

## Procedures for Workshop III

### INTRODUCTION

Use Overhead Transparency A to introduce the objectives for the workshop.

### FOLLOW-UP: TEACHING PROBLEM-SOLVING STRATEGIES

Feedback Activity

Elicit from the group a description of several of the follow-up lessons taught by the participants. Focus the discussion on the difficulties encountered by the students and on the teachers' responses to those difficulties. This discussion would be a good opportunity to reinforce the following points:

- Students learn when they encounter difficulty with the material;
- The teacher's role is to help through questions and suggestions, rather than to tell the student answers and step-by-step solution methods;
- Solving non-routine problems can advance students' learning of problem-solving strategies that are applicable to routine textbook word problems as well;
- Solving many obvious and simplistic textbook word problems does not necessarily provide meaningful problem-solving experiences for students; and
- When students verbalize their solution methods, and contrast and compare different solution methods, they are expanding their repertoire of solution strategies and their ability to decide on the appropriateness of particular strategies to a problem.

### A LOOK AT TEXTBOOK WORD PROBLEMS

Activity III. 1

Project Overhead Transparency B. Ask participants to solve the problems in their packet using the rules listed on "A Student's Misguide to Problem Solving." When they are finished with the ten problems, project Overhead Transparency C, and ask them to mark their papers.

Ask the participants for reactions to the exercise. They will notice that, if the rules were followed, the answers to most problems should be correct. The list of rules contains many misconceptions that students

develop as they deal with textbook word problems. This activity illustrates nicely the danger of relying only on written results in assessing student problem-solving skills; students may get correct answers for wrong reasons.

The first rule, "Avoid reading the problem," is perhaps the most unfortunate. In their efforts to make word problem solving easier, some teachers have inadvertently encouraged that behavior. The key word approach, a relatively common practice in problem-solving instruction, gives students the idea that word problem solving can be tackled in the same passive, rote way that they have learned to use with computational exercises. The description of a situation in a written problem requires analysis. The idea that one word or phrase can be equated with an operation leads students to search for the "key word" and forget the rest of the text.

To address this issue of key words in the discussion, use Overhead Transparency D. The list of problems which contain the traditional key words in other contexts illustrates one difficulty in the approach. The other difficulty lies in the very idea that problem solving can be reduced to a set of routine procedures.

The fact that the key word approach has inherent dangers does not mean that the role of language in problem solving can be ignored. Special instruction in reading word problems and developing appropriate vocabulary can be useful. The following points are appropriate to make during the discussion:

- The research on the relationship between reading and problem-solving ability indicates that in the upper grades, general reading difficulty is not the major factor contributing to poor problem-solving performance (Zweng, 1979; Suydam, 1980; Carpenter, et. al., 1981). Most students who incorrectly solved problems in one study could read the problems aloud and re-tell the story problems in their own words (Knifong and Holtan, 1977).

- Students lack the language skills to talk about mathematics, to communicate about problem solving; more oral language needs to be used in mathematics classrooms (Earp, 1980).

- Research indicates a relationship between measures of mathematics achievement and measures of <u>selected</u> language skills (Skypek, 1981).

- Special instruction related to the specific demands of reading mathematical problems can improve success in problem solving (Barnett, et. al., 1980).

- Some of the differences between reading mathematical word problems and ordinary prose are:

    1) Word problems are more dense than prose, that is, they contain more main ideas in one sentence or paragraph;

2) Word problems often lack the context clues that are present in prose;

3) Vocabulary differences can cause confusion. Many words have specialized meanings in mathematical problems (e.g., range, power, base, altitude, operation, prime). New terms often have no relationship to the learner's vocabulary and are therefore difficult to integrate;

4) Whereas prose usually contains a continuity of content and ideas from sentence to sentence, and paragraph to paragraph, word problems appear without this continuity. Students need to be able to quickly shift to a new context, content, and structure when solving word problems; and

5) Numerals, symbols, graphs, and charts may break a student's train of thought and require an altered reading pattern (Barnett, et. al., 1980).

- Some techniques which can be used to address these specialized reading problems are included in Activity III. 3.

Activity III. 2

Ask the participants to work in pairs to analyze a mathematics textbook they use. The activity addresses three areas: the relationship between computation and word problems, the inclusion of non-routine and multi-step problems, and the formal instruction in problem-solving. Allow between 15 and 30 minutes for the analysis.

Lead a discussion of the results of the textbook analysis. The copyright date of the text may affect the observations made about word problems. Some of the conclusions that may be made by the group are:

- Most problems, appearing at the end of a section which teaches or reviews a computational skill, apply only that particular skill to a one-step problem;

- Multi-step problems usually appear in a section labeled as such;

- Problems that require the application of a mixture of computational skills are usually labeled;

- Non-routine or challenging problems usually appear in separate sections isolated from computation and problem-solving instruction;

- Problems rarely contain superfluous or insufficient data, unless so identified; and

- The content of word problems, in older texts especially, may contain sex, ethnic or racial biases.

Whether or not the above weaknesses of textbooks are identified in the discussion, you will want to stress that the textbooks do not provide enough problem-solving experiences on their own.

## IMPROVING TEXTBOOK WORD PROBLEMS

### Activity III. 3

Review the list of suggestions for improving textbook word problems. The list contains examples of those suggestions that might not be understood without an example. Depending on the available time, you may want to solve some of the problems with the group. Using computational examples as headlines, the problem hunt, and sequencing problems for generalization would be good topics for the more in-depth treatment.

Ask participants to use their textbooks to design a problem-solving activity which implements one or more of the suggestions. Each participant should write a description of the activity. Remind the participants of the problem-solving strategies addressed in the previous workshop. They might design activities to reinforce the lessons they planned after Workshop II.

Form pairs of participants to review the activities. Ask each participant to give constructive feedback to improve his or her partner's activity. These activities will form the basis of a lesson to be taught by participants before the next workshop.

## USING OTHER RESOURCES TO GENERATE PROBLEM-SOLVING EXPERIENCES

### Activity III. 4

Assign the participants to do one or more of the three activities using the supermarket circular. Hand-held calculators should be available. Remind them to think about the strategies they use in solving the problems.

Discuss with the participants the advantages and disadvantages of the activities. Some of the issues you might address are:

- Although there are many possible answers, the recordkeeping charts in problems 1 and 2 enable the teacher to easily correct the problem;
- The hand-held calculator is an appropriate tool to have available during problem solving. You want the students to focus on the process of problem solving, rather than mechanical computation;

- The same activities can be used over again by merely changing the goal numbers; and
- Supermarket circulars and other readily available resources can be used to provide more real-life experiences (e.g., writing a shopping list for a class party or comparison shopping activities).

Activity III. 5

Distribute a copy of a newspaper to each participant. Lead a brainstorming activity with the whole group to generate a list of possible problem-solving activities using the newspaper as a resource. The resources cited in the workshop packet contain articles with such suggestions (Lindquist, 1977; Shaw, 1984).

Activity III. 6

Form groups of about five participants. Assign each group to choose and develop one idea from the generated list into a classroom activity. Ask one member of each group to take responsibility for writing a detailed description of the activity to be reproduced and shared with all participants. This could be the beginning of a resource bank of problems and activities housed in the school. Participants might want to plan a system for organizing and developing such a problem collection. Many of the commercial resources listed in the references would be useful sources to form the basis of the collection.

CONCLUSION

Assign the follow-up activity. You may want to visit the classrooms during the planned activities to assist in implementation and to gather some first-hand information about the reaction of the students to the lesson.

TRAINING MATERIALS FOR MODULE III

## OBJECTIVES FOR WORKSHOP III

The objectives of this workshop are:

1) To explore techniques for extending the word problem experiences provided by standard textbooks; and

2) To increase the ability of participants to generate problem-solving experiences from readily available resources.

# A Student's Misguide to Problem Solving

RULE 1: If at all possible, avoid reading the problem. Reading the problem only consumes time and causes confusion.

RULE 2: Extract the numbers from the problem in the order in which they appear. Be on the watch for numbers written in words.

RULE 3: If rule 2 yields three or more numbers, the best bet for getting the answer is adding them together.

RULE 4: If there are only two numbers which are approximately the same size, then subtraction should give the best results.

RULE 5: If there are only two numbers in the problem and one is much smaller than the other, then divide if it goes evenly--otherwise, multiply.

RULE 6: If the problem seems like it calls for a formula, pick a formula that has enough letters to use all the numbers given in the problem.

RULE 7: If rules 1-6 don't seem to work, make one last desperate attempt. Take the set of numbers found by rule 2 and perform about two pages of random operations using these numbers. You should circle about five or six answers on each page just in case one of them happens to be the answer. You might get some partial credit for trying hard.

RULE 8: Never, never spend too much time solving problems. This set of rules will get you through even the longest assignments in no more than ten minutes with very little thinking.

(This was taken from the North Carolina State Math Newsletter. The author, Joe Dodson, is Mathematics Supervisor for the Winston-Salem/Forsyth County Schools.)

## ANSWERS

1) 54

2) 3

3) 16

4) $59.50

5) $ 7.15

6) 158

7) $ 5.86

8) 147

9) 6

10) 54

## KEY WORDS?

1) Kimio had 15 cookies left after his friends ate 13. How many did he have before they ate the cookies?

2) Joan had some marbles in her pocket. Marta gave her 16 more. Joan had 25 altogether. How many did she have in her pocket?

3) Ricardo shared 36 pennies among 7 children so that each child had a different amount. How many might each child get?

4) Jackie missed school 15 times during the year. She was sick 1/3 of the time. What part of the time was she not sick?

5) Raquel walked 18 blocks north and turned left. She then walked east. Altogether she walked 13 blocks. How many more blocks did she walk north than east?

# MODULE IV

# PROBLEM SOLVING USING MODELS AND MANIPULATIVES

PACKET FOR WORKSHOP IV

# Problem Solving Using Models and Manipulatives

New Jersey State Department of Education
Division of Compensatory/Bilingual Education
Spring, 1985

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## INTRODUCTION TO WORKSHOP IV

> A great discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery. Such experiences at a susceptible age may create a taste for mental work and leave their imprint on mind and character for a lifetime. (Polya, Preface)

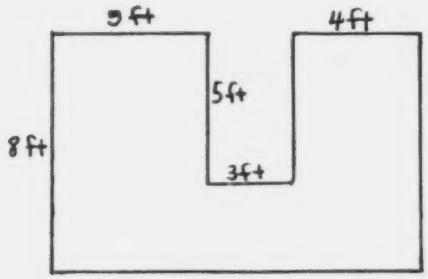
Good problem solvers have a "taste for mental work," that is, they enjoy engaging in problem solving. Teachers of problem solving want to find ways to encourage this desire to solve problems and the confidence to persevere in the face of difficulty. Unfortunately, some students in the middle grades have experienced years of failure and boredom with mathematical problem solving.

Problem-solving experiences using models and manipulatives can help students feel again the "triumph of discovery" in a new context. Many of the same problem-solving strategies and skills that are applicable to word problems can be developed through the use of models and manipulatives to pose problems and to represent solution processes. Some computer software is available that encourages the use of problem-solving strategies and skills (e.g., Rocky's Boots, Gertrude's Puzzles, and Moptown Hotel from The Learning Company; The Factory from Sunburst Communications). When students lack previous negative associations, they are more likely to become involved in the problem-solving process. If students are encouraged to verbalize, and thus become conscious of their thinking processes in these non-traditional contexts, they can learn to apply these same strategies and skills to all problem solving.

The motivational issue is not the only justification for involving students in more problem solving with models and manipulatives. The experience of representing skills and strategies in more than one mode, that is, concretely with objects, iconically with pictures and symbolically with numerals deepens understanding of these skills and strategies (Bruner, 1966, pp 10-11; Wirtz, 1976, pp 1-5). Students who have weak computational skills can nevertheless engage in meaningful problem-solving experiences.

While students engage in problem-solving experiences with manipulatives and models, they have the opportunity to build their understanding of other basic skills. For example, the geoboard activities included in this workshop can be used to meaningfully teach concepts of area and perimeter. Consider the following item on the New Jersey High School Proficiency Test (Spring, 1984).

What is the area of this figure? (All angles are right angles.)




The student who has merely memorized a formula to find the area of a rectangle may not be able to answer such a question correctly. Experience in finding the area of irregularly shaped figures on the geoboard may enable students to develop a deeper understanding of area concepts.

In addition, practice with the manipulation of objects and pictures of these objects is key to developing spatial visualization abilities. The skills involved in spatial visualization, the ability to manipulate objects mentally, are essential to success in mathematics and science. Geometry, in particular, requires one to picture how an object would look, if it were flipped, rotated, or sliced down the middle.

The inclusion of more activities that teach spatial visualization may have an impact on sex-related differences in mathematics achievement. Researchers have hypothesized that the ability of boys to outperform girls on tasks that require spatial visualization may be a factor in accounting for the higher mathematical achievement of boys that begins in the adolescent years (Fennema, 1977; Fennema and Carpenter, 1981; Fennema and Sherman, 1977; Maccoby and Jacklin, 1974; Sherman, 1967).

This workshop will suggest problem-solving activities using manipulative materials such as Cuisenaire Rods, Geoboards and Pattern Blocks. In addition, two pictorial languages from the Comprehensive School Mathematics Program, CSMP (1978) -- the languages of strings and arrows -- will be introduced to illustrate the use of models for posing and solving problems.

## OBJECTIVES FOR WORKSHOP IV

The objectives for this workshop are:

1. To explore problem-solving activities with manipulative materials; and

2. To examine the use of mathematical models to pose and solve problems.

## FOLLOW-UP: EXPANDING WORD PROBLEM-SOLVING EXPERIENCES

### Feedback Activity

Share with the group your reactions to the activity you planned and implemented after Workshop III. Focus on how you plan to improve what you did and what other similar activities you would like to try. Take notes on the ideas you get during the discussion.

## PROBLEM SOLVING WITH MANIPULATIVES

Activity IV.1

Spend a few minutes becoming familiar with Cuisenaire Rods. Follow Polya's four-step strategy to solve this problem with Cuisenaire Rods:

> How many different 3-rod trains can you make that are equal in length to a brown rod?

Step 1: Understanding the Problem

Step 2: Devising the Plan

Step 3: Carrying Out the Plan

Step 4: Looking Back

Discuss the various specific strategies that can be used to solve the problem.

Activity IV.2

Work with a partner. Spend ten to fifteen minutes problem solving with each manipulative material. Practice "talking mathematics"; verbalize your thinking to your partner. Discuss the problem-solving strategies you are using. When your partner experiences difficulty, do not show or tell him or her the answer. Try to ask questions that will assist him or her to look at the problem in a new way.

Keep notes on your experiences.

| Name of Material/Activity | Reactions |
|---|---|

TOOTHPICK PUZZLES

1. Use 17 toothpicks to construct this figure.

a. Remove 5 toothpicks and leave 3 squares.
b. Remove 6 toothpicks and leave 2 squares.

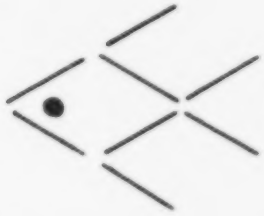
2. Make this figure with 12 toothpicks.

a. Remove 4 toothpicks and leave 3 triangles.

3. With 9 toothpicks, make this figure.

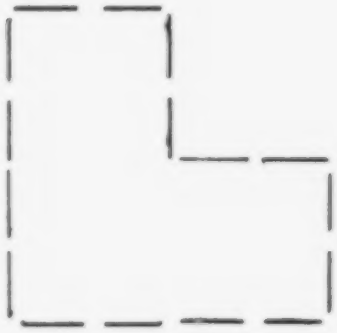
a. Remove 2 toothpicks and leave 3 triangles.
b. Remove 3 toothpicks and leave 1 triangle.
c. Remove 6 toothpicks and leave 1 triangle.
d. Remove 4 toothpicks and leave 2 triangles.
e. Remove 2 toothpicks and leave 2 triangles.

4. Use 8 toothpicks and 1 button to form a fish.



Move 3 toothpicks and the button to make this fish swim the opposite direction.

5. Two farmers have land this shape.



a. The first farmer wants to divide her land evenly among her three daughters. Add 4 toothpicks to form three parcels of equal size and identical shape.

b. The second farmer wants to divide her land evenly among her 4 daughters. Use 8 toothpicks to form four parcels of equal size and identical shape.

6. Use 6 toothpicks to form 4 equilateral triangles.

7. Make up two toothpick puzzles of your own, with a story problem. Try them out with your students.

PATTERN BLOCK PROBLEMS

Exploration

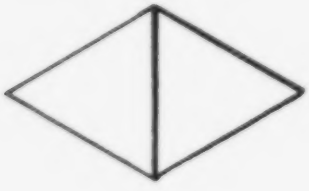
Can you cover up a flat space (make a floor) with just hexagons?...triangles?...squares?...trapezoids?

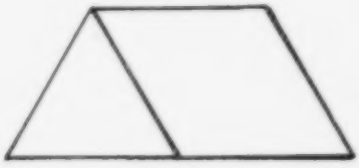
Make squares from squares. How long can you keep going?

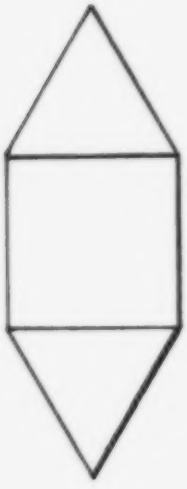
How many different ways can you make the yellow hexagon from the other blocks?

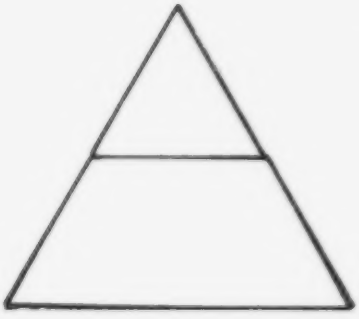
Structured Problems

Cover the shapes with blocks.

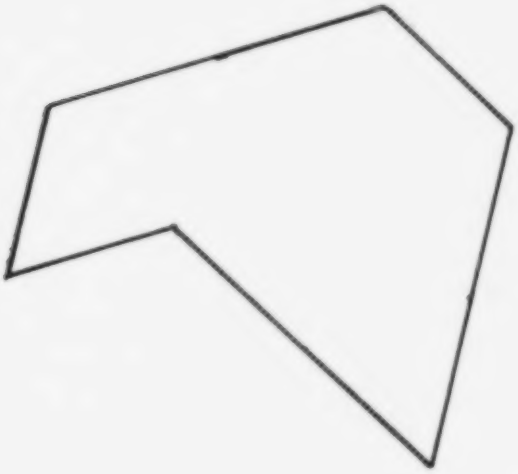




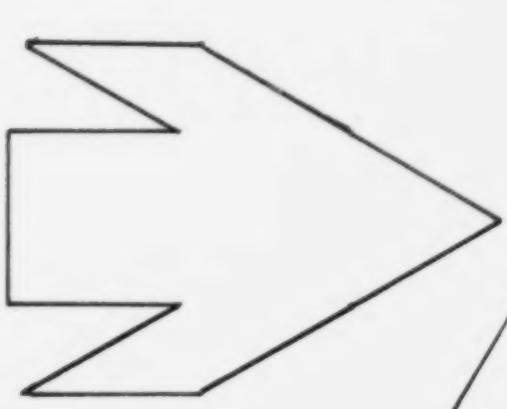


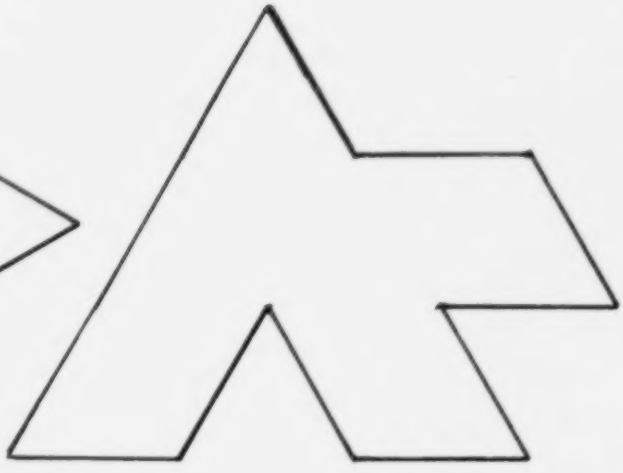


Cover each of these figures with exactly three blocks.

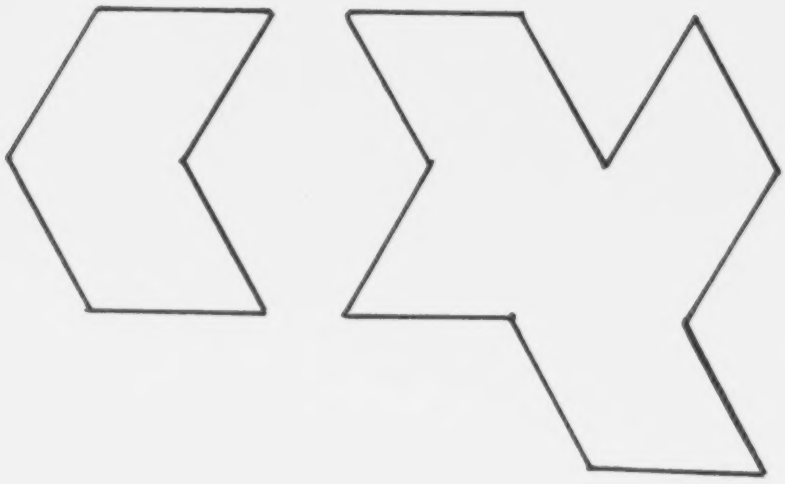


Cover each of these figures with exactly five blocks.

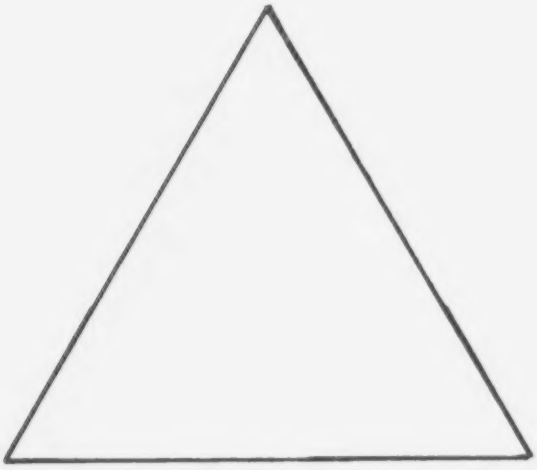




Cover each of these figures with the fewest blocks that you can.

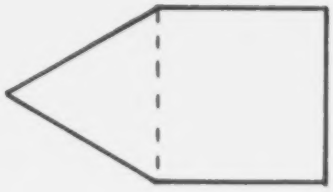


Cover this triangle in four different ways using one red trapezoid, two blue rhombi (wide diamonds), and two green triangles.



## A Polygon Problem with Pattern Blocks

The green triangle is a 3-sided polygon. The orange square, the blue rhombus, and the tan rhombus are 4-sided polygons. This is a 5-sided polygon:



The yellow hexagon is a 6-sided polygon. Here is another one:

All of these polygons are equilateral, that is they have equal sides, but they are not necessarily equiangular (with equal angles). If they did have equal sides and equal angles, they would be regular polygons.

All of these polygons are convex, that is, the sides don't bend in.

Use the Pattern Blocks. Can you make polygons of 7 sides? ...8? ...9? ...10? ...11? ...12? or more sides which are convex and equilateral. They need not be equiangular.

Can you make any of them in more than one way?

This problem was adapted from Elementary Science Study, Teacher's Guide to Pattern Blocks.

## TANGRAM PUZZLES

Exploration

What can you make with tangram pieces?

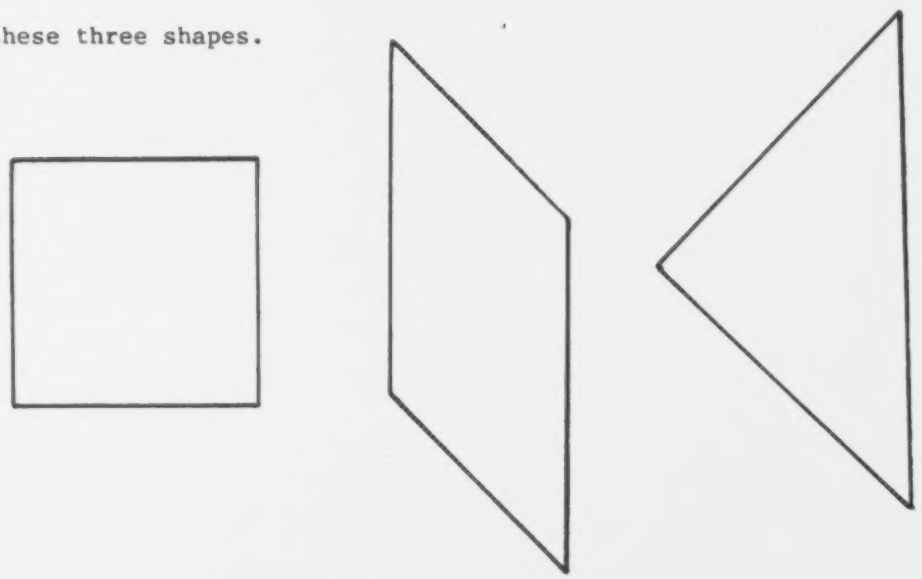
Which pieces can you make out of other pieces in the set?

Order the pieces from smallest area to largest area. Justify your order.

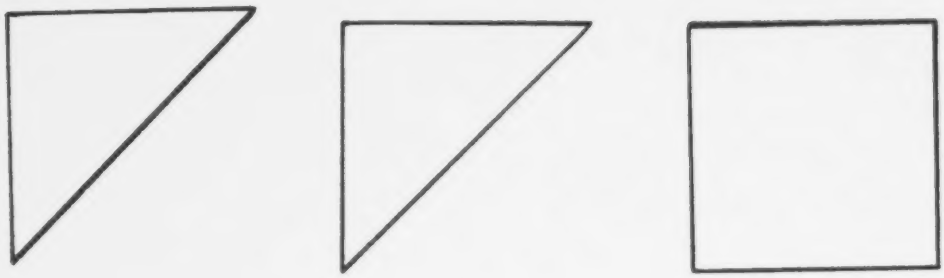
Structured Problems

With these two pieces,

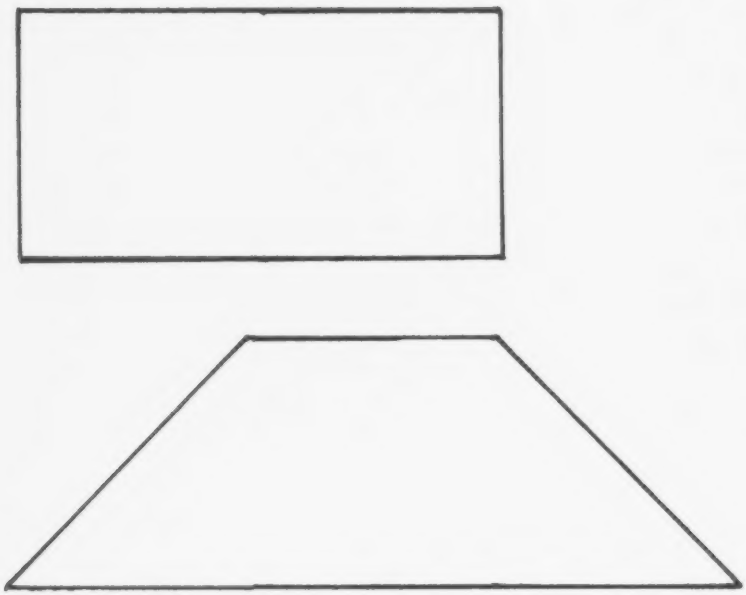
make these three shapes.



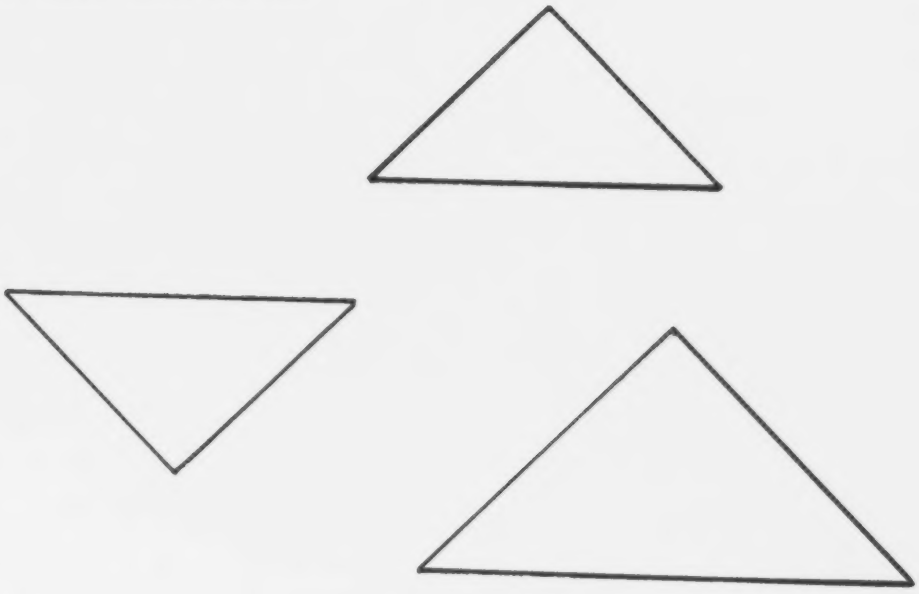
With these three pieces,



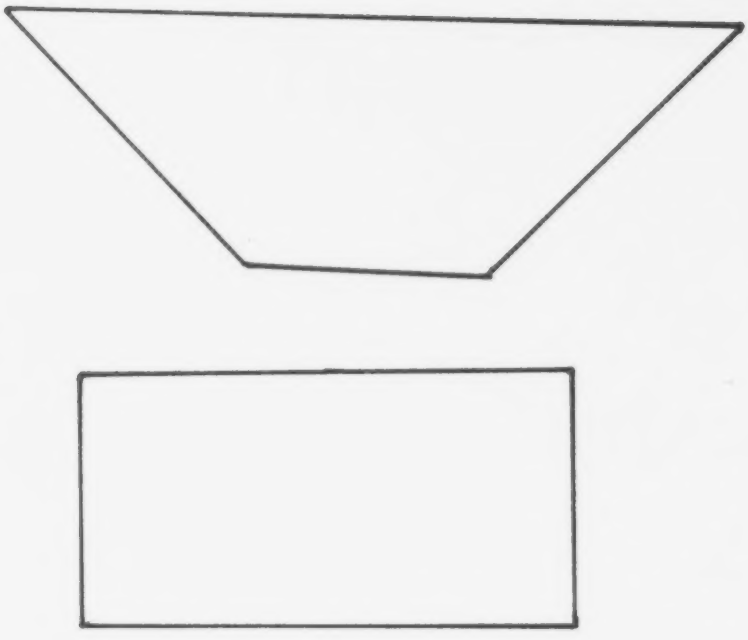
make these two shapes.



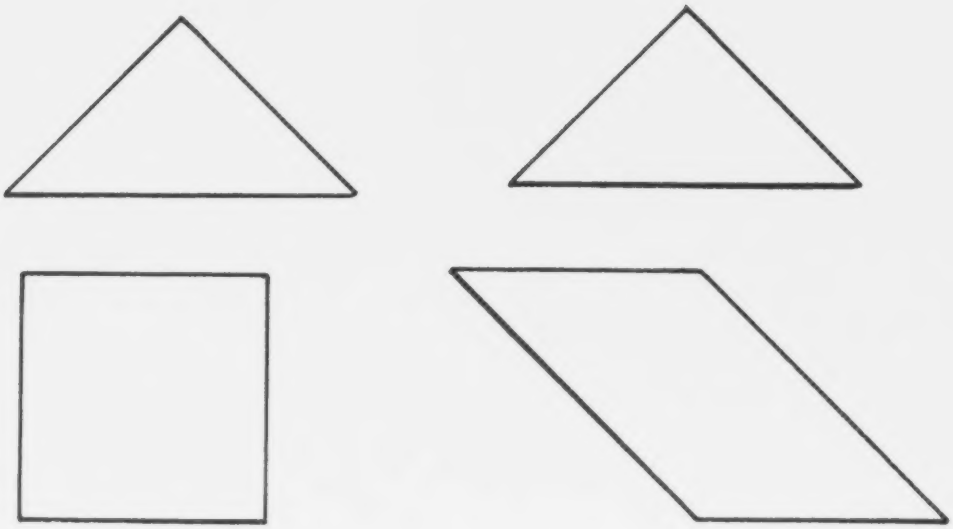
With these three pieces,



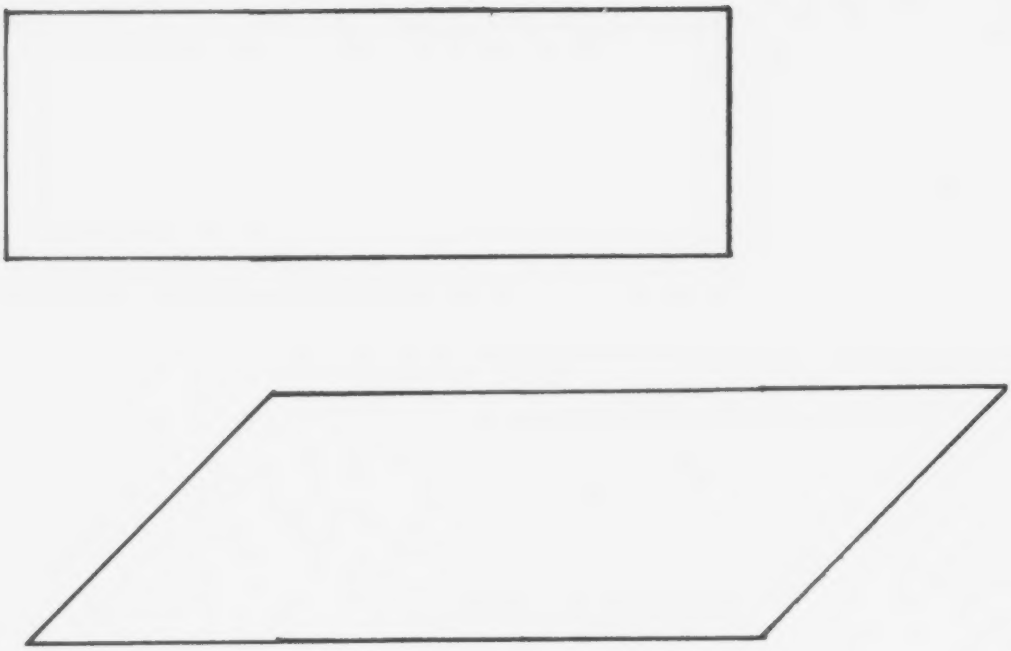
make these two shapes.



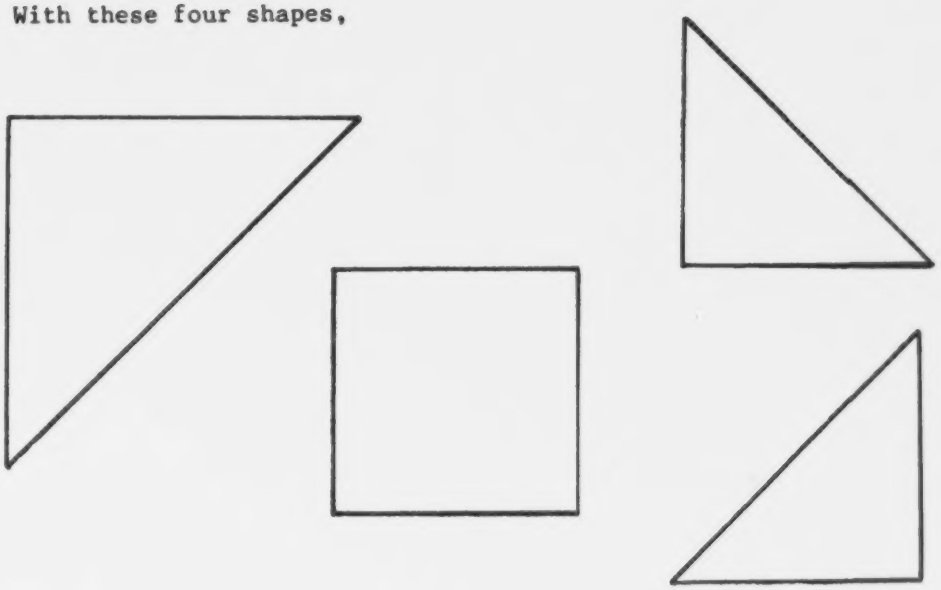
With these four shapes,



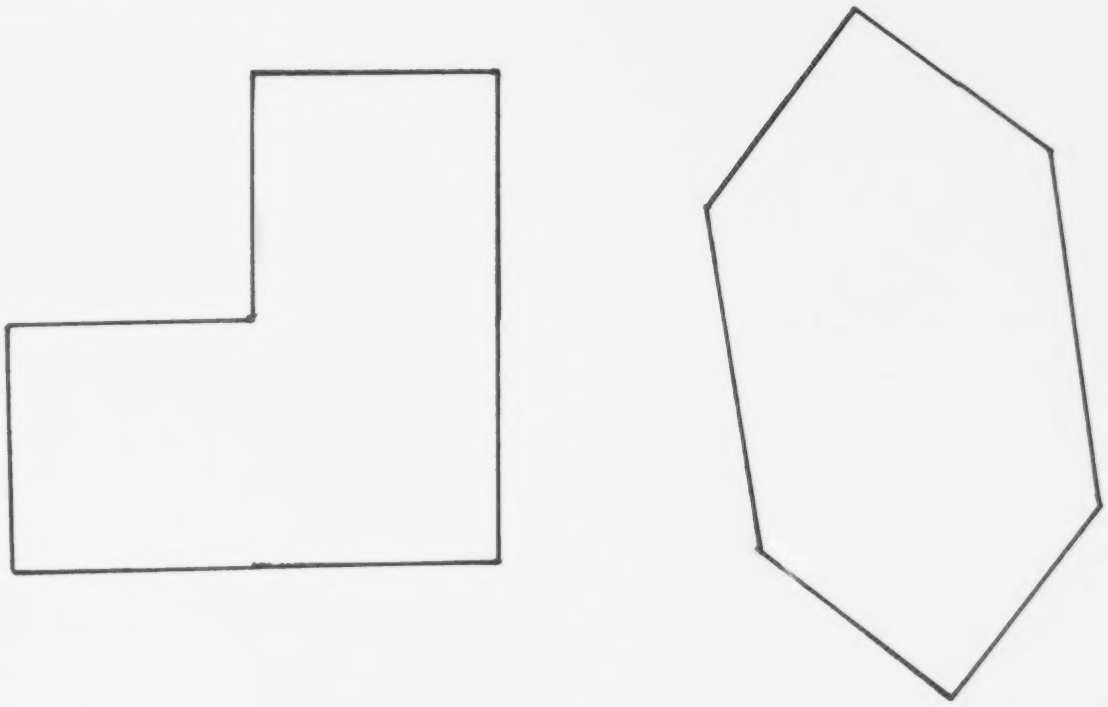
make these two shapes.



With these four shapes,



make these two shapes.



For a real challenging problem, try to make a square using all seven tangram pieces. If you cannot get a solution right away, don't get discouraged. Try again later.

How many other convex polygons can you make using all seven pieces?

Make up some tangram puzzles of your own to try out with your students.

## PROBLEMS ON THE GEOBOARD

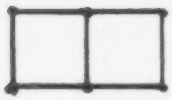
Exploring Perimeter and Area

is called 1 unit.

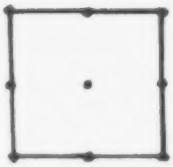


is called 2 units.

The distance around the square or perimeter is 4 units. The area is 1 square unit.

1) 

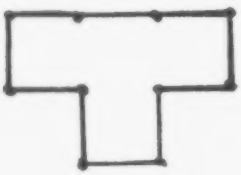
The area is 2 square units.
The perimeter is _____ units.
The number of nails touched is _____.

2) 

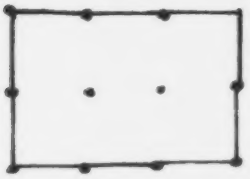
The area is _____ square units.
The perimeter is _____ units.
The number of nails touched is_____.

3)

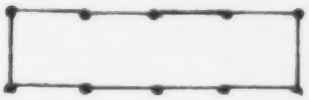
The area is _____ square units.
The perimeter is _____ units.
The number of nails touched is ____.

4) 

The area is _____ square units.
The perimeter is _____ units.
The number of nails touched is _____.

5) 

The area is _____ square units.
The perimeter is _____ units.
The number of nails touched is _____.
The number of nails inside is _____.

6) 

The area is _____ square units.
The perimeter is _____ units.
The number of nails touched is _____.

7) Look at exercises 2 and 3 above. Can the perimeters be the same and the areas vary? _____

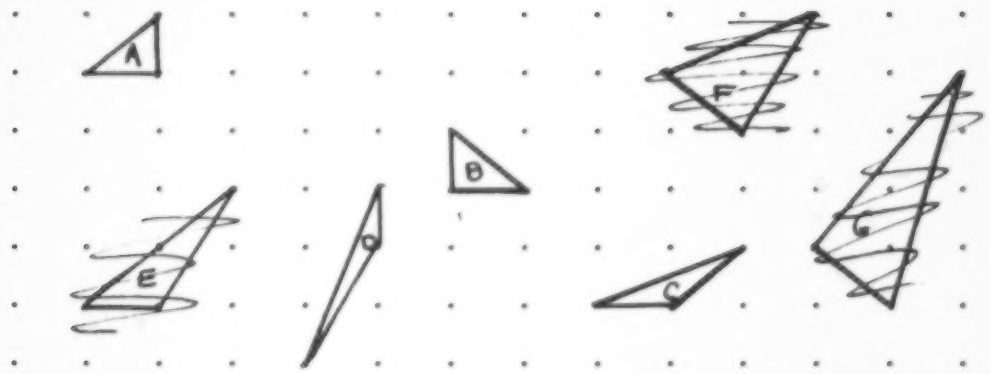
8) Which has the most area with the same perimeter, a square or a rectangle? _____

9) After you have made three different shapes on the geoboard with the perimeter of 12 units, draw the shapes on dotted paper.

10) Make at least five different shapes with an area of 2. What is the perimeter of each? Record your results on dotted paper.

11) Make a figure with an area of 6 square units and a perimeter of 14 units on your geoboard and record your results on dotted paper.

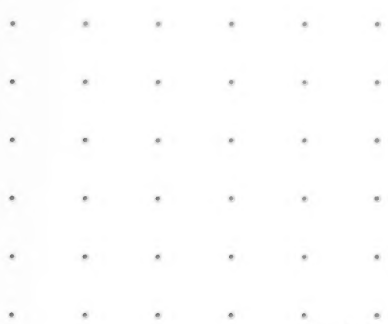
Pick's Theorem on the Geoboard




What do you make of this doodling? All are triangles, but two were erased. Why? What's the "3d (0)"? How are A, B, C, and D alike, but different from the ones erased?

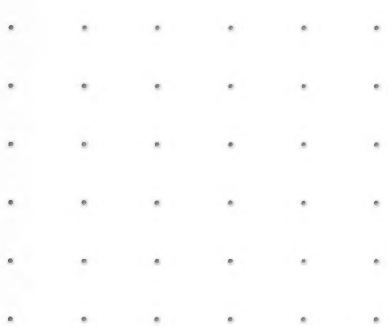
3d (0) is a shorthand for a "3 dotter with 0 dots inside." So one triangle was erased because it touched 4 dots, and the other, because it had 1 dot inside.

Now it's your turn to doodle. Make several different shapes that belong to each group.

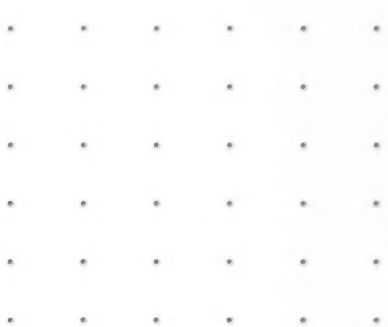


4d (0)

4d (1)

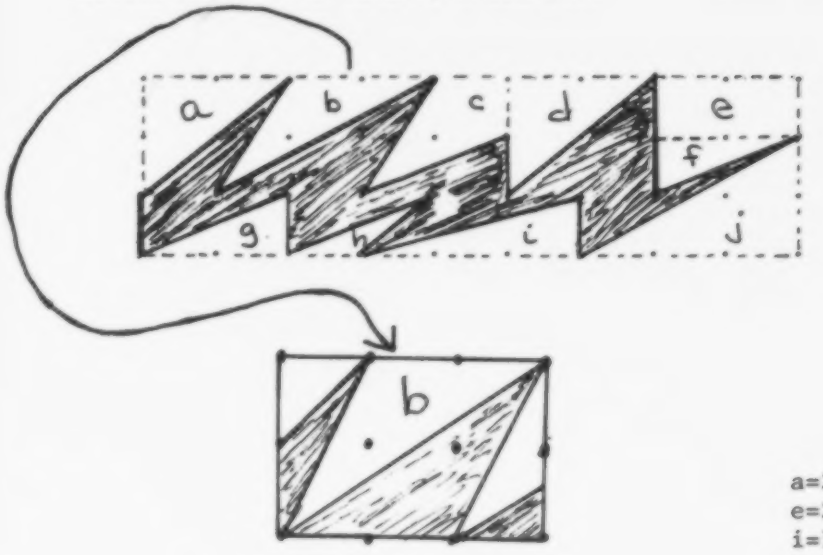


4d (2)



3d (1)

Can you find the area of this streak of lightning? This streak is a 20 dotter with 1 dot inside, a 20d (1).




a=2 b=2 c=2 d=2
e=2 f=2 g=1 h=1/2
i=1 1/2 j =3

a + b + c + d + e + f
+ g + h + 1 + j =

The area of the big rectangle around the lightning is _____ units. The part that is shaded is _____. So, the area of the shaded part must be _____ units.

Now perhaps you have some ideas that will help you find the area of any doodle on the previous page.

Here are some extra grids to help.

In each section draw at least two of the shapes indicated by our doodler's shorthand. Find the area of each by any method you like. You probably are already convinced that all shapes described by the same piece of shorthand have the same area.

---

3d (2)

area =

---

7d (4)

area =

---

Complete this table doing any experiments you need to do.

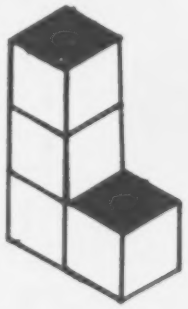
| boundary dots | (0) | (1) | (2) | (3) | (4) | (5) |
|---|---|---|---|---|---|---|
| 3 | 1/2 | | | | | |
| 4 | | | | | | |
| 5 | | | | | | |
| 6 | | | | | | |
| 7 | | | | | | |
| 8 | | | | | | |
| 9 | | | | | | |
| 10 | | | | | | |

Do you notice a pattern?

If we use $b$ to represent the number of dots on the boundary, and $i$ as the number of dots inside, can you write an expression for finding areas?

## THREE-DIMENSIONAL PROBLEM SOLVING WITH CENTIMETER CUBES

Make this building out of centimeter cubes.

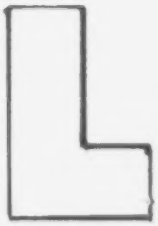


A two-dimensional drawing of four faces of the building could be drawn as follows:

TOP

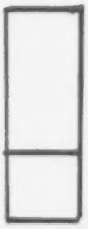
If you look down on the building from the top, you see two squares at two different levels. The levels are indicated by the line drawn in the diagram between the two squares.

FRONT



If you look at the building from the front, you see an "L" made of 4 squares all at the same level. No lines are drawn between the squares.

RIGHT



If you look from the right, you see 3 squares, 2 on one level and 1 on another level.

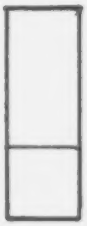
LEFT

If you look from the left, you see 3 squares, all on the same level.

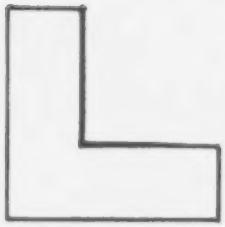
Make a building to match each set of two-dimensional drawings.

A. TOP

RIGHT

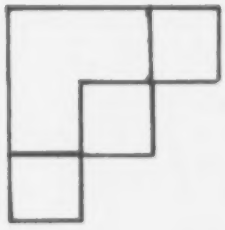


FRONT

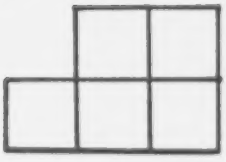


LEFT

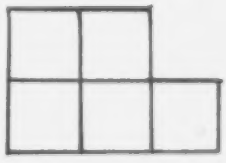
B. TOP



RIGHT

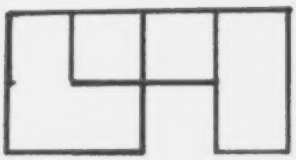


FRONT

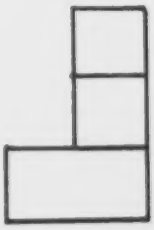


LEFT

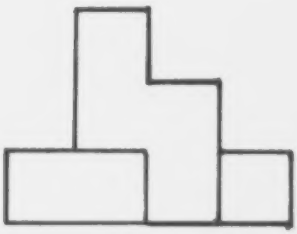
C. TOP



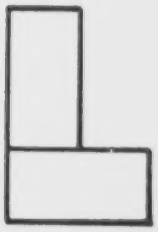
RIGHT



FRONT



LEFT



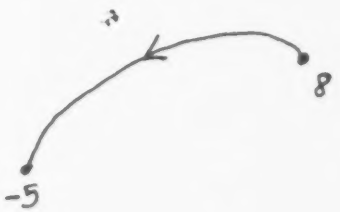
D. Now make your own building and draw the faces in two-dimensions for someone else to use to reconstruct the building.

## Activity IV.3: The Language of Arrows

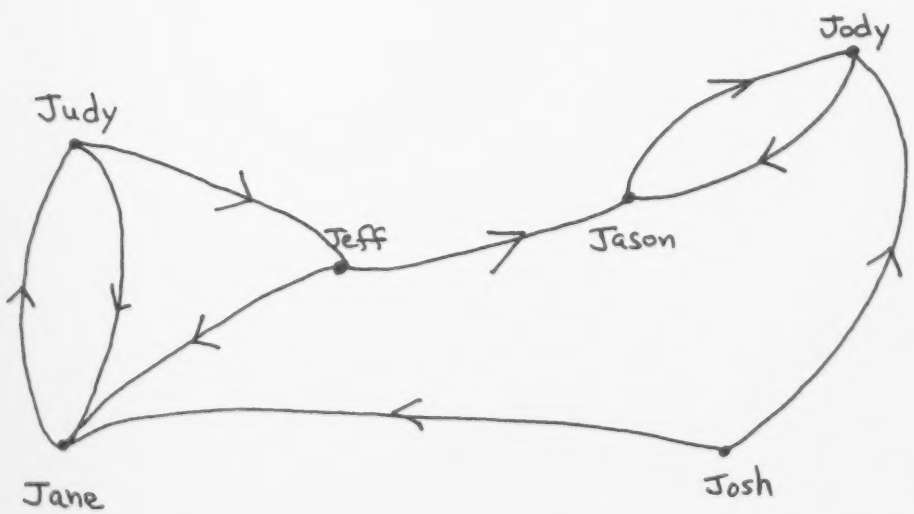
The language of arrows provides a model to represent processes involved in comparing, analyzing, and operating on sets, that is, the mathematical ideas of functions and relations (CSMP, 1978, pp ix-xi). The arrow provides a pictorial way of representing even complex relationships.

Examine the following examples of arrows and cords (connections without directionality):

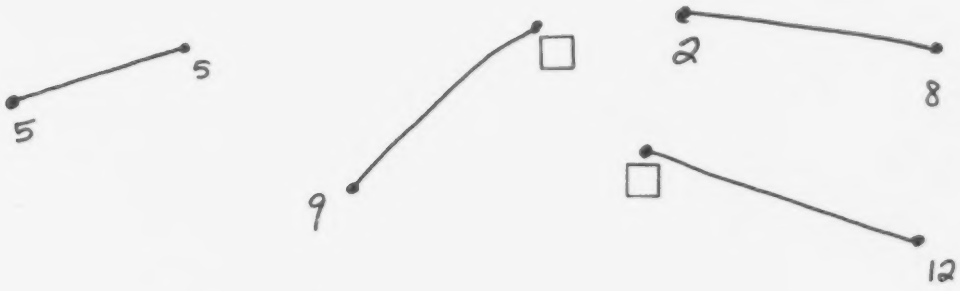
A. means "is greater than."



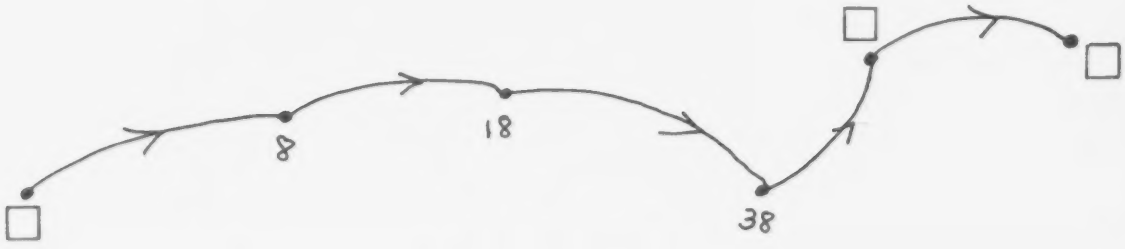

B. means "you are my friend."



C. means "sums to 10."




D. means "double and add 2."




Comment on each picture. What can you say about the relationships expressed? What numbers should go in the empty boxes?

FRIENDS

The dots represent the students in a class. Use your blue pencil to copy the arrows from the chalkboard.

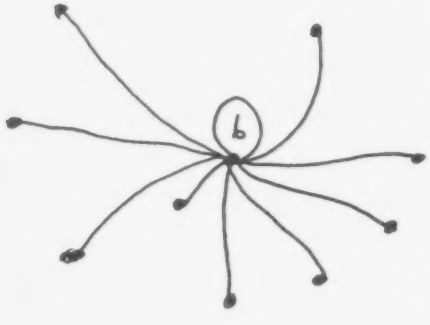
What can you say about the students?

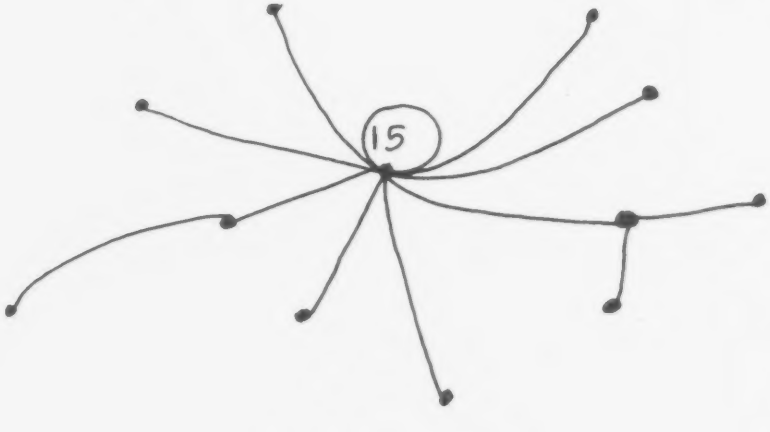
The dots represent me and my family - my brothers, sisters, parents, grandparents, cousins, uncles, and aunts. Where am I in this picture?

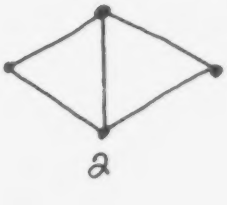
Notes

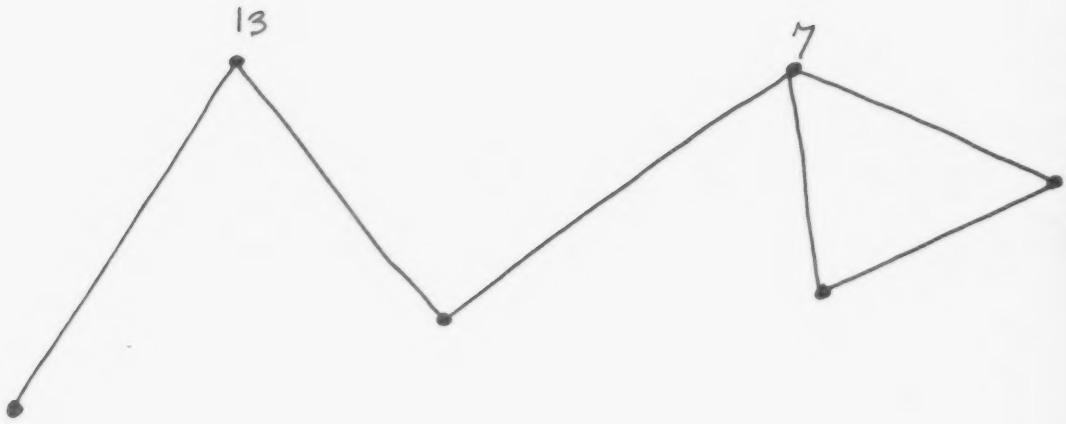
The Rule: Two numbers can chat with each other, if and only if one number is a factor of the other.

Fill in possible labels for all the dots:






The rule does not allow 15 and 6 to chat. How could they send messages to each other?

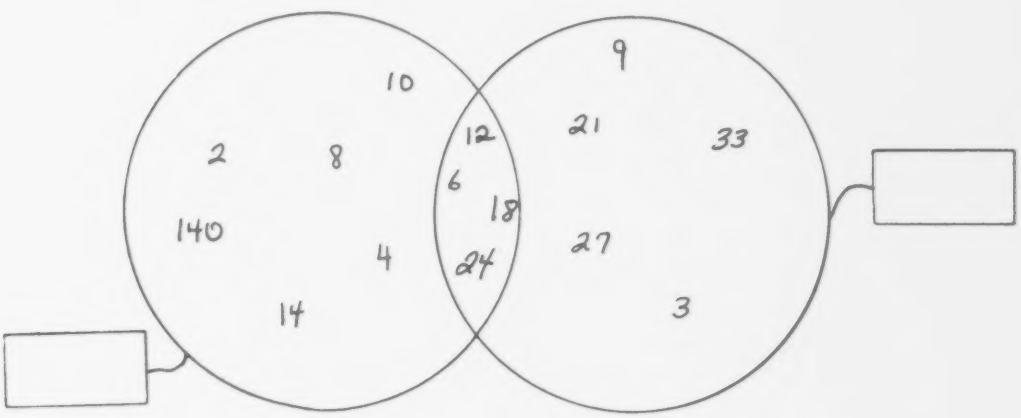
Make up your own rule and your own problems with chatty numbers.

Activity IV.4: The Language of Strings

The language of strings is a model for organizing and collecting data, that is, it deals with the mathematical notion of sets (CSMP, 1978, pp vi-xi). Strings provide a pictorial means of recording and communicating information about classification. Objects to be sorted can be represented by dots, and sets into which they are sorted are represented by drawing "strings" around the appropriate dots. A dot inside the string represents an object in the set; a dot outside the string represents an object that is not in the set.

In determining situations that make good use of strings and dots, it is important to ask students to classify objects which have well-defined attributes. For example, if students were asked to draw a string picture representing all tall people in the class, there would be disagreements about who was tall and who was not. The discussion would focus on whether someone had the attribute rather than on how to classify by that attribute.

Attribute Blocks and sets of numbers are appropriate for posing classification problems because their attributes are well-defined. Consider the following string picture.




What should the labels for each string read? Put some more numbers in the strings.

The string games in which you will participate are structured as team games. Your team will compete together against another team. As you engage in these activities, consider the advantages and disadvantages of this way of organizing a lesson.

## USING NEW IDEAS

Summary Activity

Think about all four workshops. Jot down ideas, problems, activities, and/or approaches that you plan to use in your classroom as a result of the training.

Identify the assistance, if any, that you need in implementing these ideas. What materials do you need? How can the curriculum leader in your school support your efforts? How could future in-service periods be used most effectively to improve problem-solving instruction?

Discuss your observations with the trainer and the other participants.

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WORKSHOP PRESENTER'S GUIDE

Module IV: Problem Solving Using Models and Manipulatives

## Trainer Overview of Module IV

WORSHOP OBJECTIVES:

1) To explore problem-solving activities with manipulative materials; and

2) To examine the use of mathematical models to pose and solve problems.

TRAINING MATERIALS:

Workshop Packet IV
Dot paper and Grid Paper
Copies on appropriately colored card stock of Pattern Blocks, Attribute Blocks, Attribute Cards, and Tangram Pieces for each participant
Centimeter Cubes
Pattern Blocks
Cuisenaire Rods
Tangrams
Geoboards and Rubber Bands
Set of Cut Out Paper Attribute Blocks and Cards,
Tape or Magnetic Material
Colored Chalk
Red and Blue Pencils for Each Participant
Chalkboard
Toothpicks

PREPARATION SUGGESTIONS:

- Read the procedures and anticipate questions that may be asked by the participants.

- Become familiar with the manipulative materials and models used in the workshop.

- Solve all the problems in the packet.

- Try out using some of the manipulatives and problems with students.

- Reproduce copies of the paper Attribute Blocks, Tangrams, and Pattern Blocks on appropriately colored card stock. If you need to use the paper materials in the workshop, you will need to have them cut out; otherwise, the materials are designed for teachers to use in their classrooms.

- Prepare the overhead transparencies and sufficient copies of the workshop packet.

Think about follow-up actions that you might take after the workshop to assist teachers in implementing the ideas presented in the four workshops.

## Procedures for Workshop IV

INTRODUCTION

Use Overhead Transparency A to introduce the objectives for the workshop.

FOLLOW-UP: EXPANDING WORD PROBLEM-SOLVING EXPERIENCES

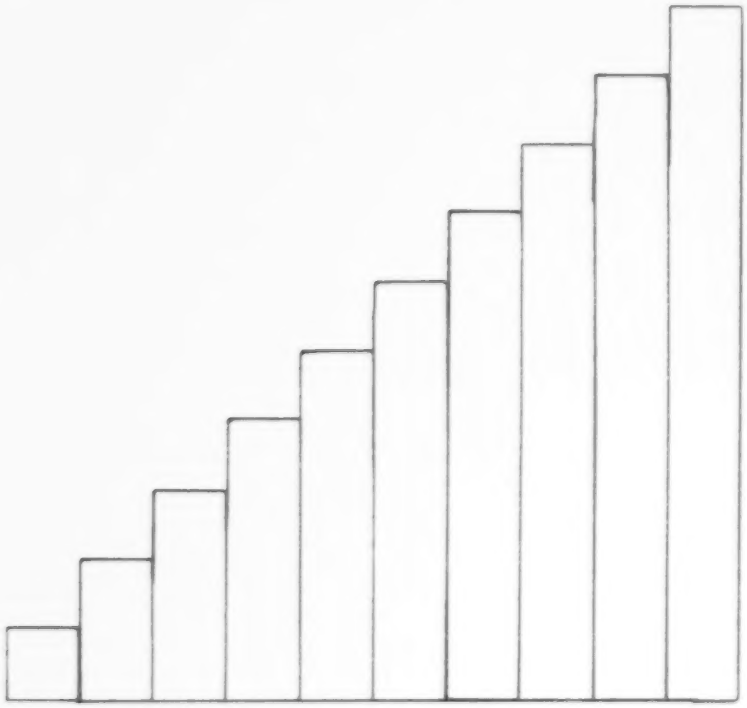
Feedback Activity

Discuss the lessons conducted by the participants after the last workshop. If you had the opportunity to visit classrooms during the lessons, share your reactions in a positive, constructive way. Highlight one or two of the activities and elicit suggestions for improvement and follow-up. You may want to develop a mechanism to ensure the sharing of problems and activities among the participants beyond the time of the workshop.

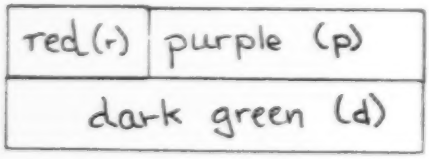
PROBLEM SOLVING WITH MANIPULATIVES

Activity IV. 1

Distribute a set of Cuisenaire Rods to each participant. Allow several minutes for free exploration of the rods. If the participants have never used the rods before, you should ask them to make a staircase of rods from the smallest to the largest to use for reference. If they are familiar with the rods, they will already know the size relationships of them.

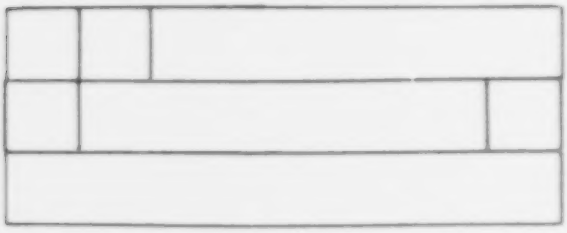


Explore the problem as a whole group. You will need to show new users what it means to "make a train". The following is a 2-rod "train" of red plus purple, equal in length to dark green:




r + p = d

White plus white plus dark green or white plus dark green plus white are two different 3-rod trains equal to brown.



w + w + d = n

w + d + w = n

You may want to have participants indicate their understanding by making another 3-rod train equal to brown.

Ask participants to work with a partner to devise a plan, carry it out and look back. Remind them to verbalize their thinking during problem solving and to refrain from telling answers.

Discuss with the whole group the strategies used to tackle the problem. Some possible approaches that might be used follow:

- Each of the possible 21 trains could be physically made with the rods. An organized, systematic procedure would be needed to ensure that all possibilities were considered.

- An organized list of all the possible trains might be made using letter symbols for the rods. For example, w + w + d, w + d + w and d + w + w are the three possible trains using 2 whites and dark green; here you are dealing with subproblems.

- The problem might be done by considering a simpler case and generalizing. For example, how many 3-rod trains could be made that are light green long? purple long? yellow long? Is there a pattern that is developing that will help you predict how many can be made that are brown long? Can the pattern be used to predict how many could be made that are orange long? orange plus yellow long? Can you write a formula?

Be sure to help the group consider the various alternate approaches to the problem. What strategies could work and which ones are not appropriate? You will also want to lead the participants in posing other similar problems that flow from the original problem. For example, what happens if 4-rod trains are made?

Point out during the discussion that this problem can be solved by strategies which can be expressed concretely with objects, iconically with pictures, or abstractly with symbols. Even the problem, which was posed in terms of the manipulation of objects, could be expressed symbolically as "How many different ways can you select three natural numbers such that their sum is 8?" Then, an organized list strategy might be recorded as follows: (1,1,6), (1,6,1), (1,2,5,), (1,5,2)...

Stress that the use of manipulative materials for problem solving provides opportunities for students with weak computational skills, or poor reading and writing skills to develop problem-solving strategies and skills. Thus, low-achieving students can experience success.

Activity IV. 2

To prepare for this activity, place each manipulative material on a table or area of a table with chairs arranged around the table. The participants will be moving from table-to-table to try as many of the activities with the different manipulative materials as time permits. At least 30 minutes should be allocated to this exploration.

Participants should work in pairs. They should practice using good questioning techniques to help each other without giving away the answers. Particularly in solving the toothpick puzzles and the activities with Pattern Blocks and Tangrams, the participants will need to "break set" between problems. That is, they will need to look at the problems from different points of view.

Encourage them to write notes about their experiences with each material.

Circulate during the problem-solving period. Model good instructional behavior by asking appropriate questions and encouraging thinking aloud. Gather examples of good questioning techniques and strategies that you observe as you interact with the pairs.

Warn the group five minutes before you will bring them back together. You may want to point out that some of them may be frustrated because the time is limited. They will want to remember this in designing activities for their students. Ideally, learning to problem solve should not be a timed activity.

Discuss the reactions of the participants to the activities. The following points should be integrated into the discussion:

- Manipulatives, that is, concrete objects that can be acted upon by the learners, can be used as models for mathematical concepts. This distinction between a manipulative and a model can best be made with several examples related to teaching computation:

1) Cuisenaire Rods are manipulative materials. They can be used to model whole number addition by assigning the value of one to the white rod. So, the number sentence 8 + 2 would be represented by a train of brown and red which is orange long. If Cuisenaire Rods are used as a model for division of fractions, the orange rod could be assigned the value of one and the number sentence 4/5 - 1/10 could be expressed by finding how many white rods (1/10) there are in the brown rod (4/5).

2) Many models can be used to represent addition of fractions. For example, if the yellow hexagon from the Pattern Blocks is assigned the value of one, 5/6 + 1/3 could be represented by adding five green triangles to one blue rhombus (a parallelogram with equal sides). The sum could then be represented as one yellow hexagon (1) and one green triangle (1/6) or 1 1/6. If the dark green Cuisenaire Rod is assigned the value of one, the same example would be represented as a yellow rod plus a red rod.

Not all models that can be used to develop concepts are manipulative materials. The next two activities will introduce pictorial languages, from CSMP, the Comprehensive School Mathematics Program, a nationally validated program developed by CEMREL, an educational laboratory; they are powerful models for a variety of mathematical concepts. Logo is an accessible, yet sophisticated computer language that involves students in meaningful mathematical explorations (Abelson and di Sessa, 1980; Papert, 1980).

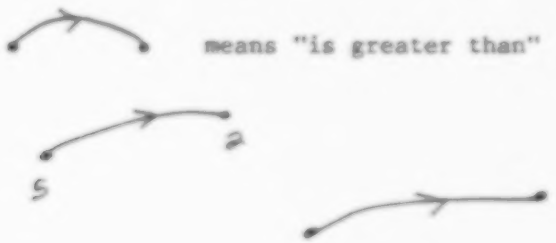
- Manipulatives and models serve an important role in the development of abstract concepts (Driscoll, 1980). Abstraction in mathematics is a slow process. The teacher helps students bridge from the concrete to the abstract. Experiences with a variety of models and manipulatives, that is, multiple embodiment, during concept formation encourages students to make abstractions, and to do so without associating the concept uniquely with a particular model (Dienes, 1970). For example, if the geoboard is the only material used to model area concepts, students might incorrectly abstract from the situation that area only applies to plane figures with straight sides that can be constructed on the geoboard.

- Spatial visualization skills are involved in doing all the activities. These are skills that are highly correlated with mathematics achievement (Fennema and Sherman, 1977). Students who engage in activities designed to highlight these skills improve in their ability to visualize three-dimensional objects and to record these objects in two-dimensions (Lappan and Winter, 1982).

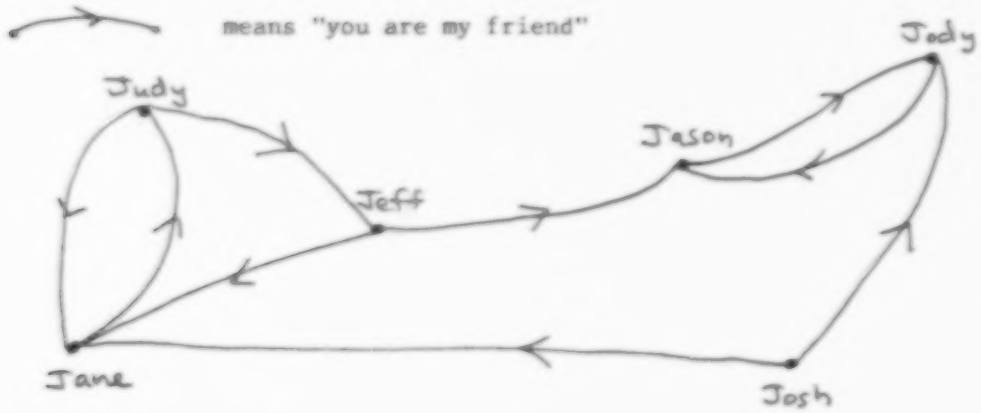
PROBLEM SOLVING WITH PICTORIAL LANGUAGES

Activity IV. 3

Distribute red and blue pencils to the participants. Draw two dots labeled as shown below on the chalkboard with white chalk. Draw an arrow between the dots with blue chalk. Begin to draw the arrow by pointing to the starting dot, "5", and saying as you draw it, "5 is greater than 2."

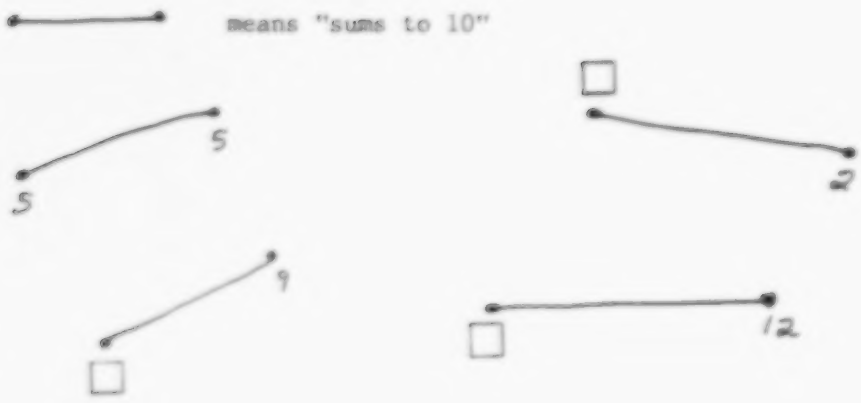



Draw two unlabeled dots on the board and draw a blue arrow between them. Ask participants to suggest possible labels for the dots. Now erase the board and indicate that you are changing the rules. Introduce a new blue arrow that means "you are my friend." Draw the following diagram on the board.



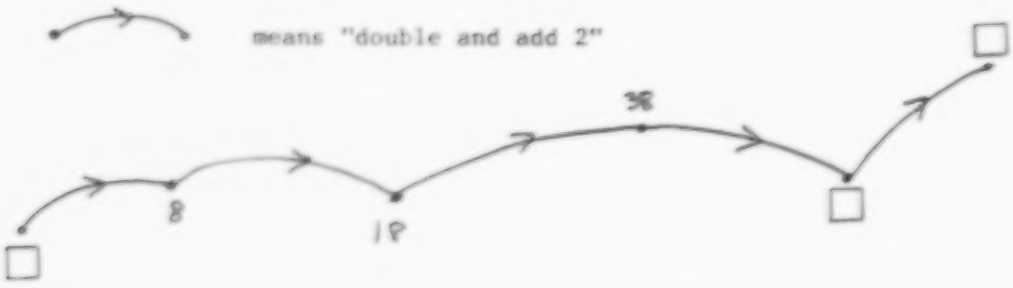

Ask the participants to tell you about what they know about these children. For example, Jeff says to Jason, "You are my friend." Does every child have someone who calls him or her friend? Are there any mutual friendships? Who has the most children calling him or her friend?

Erase the board and introduce a cord that means "sums to 10". Draw the following picture:




Have the participants fill in the empty boxes to label the dots appropriately. You should point out the difference between the arrow and the cord, that is, the cord has no directionality and the arrow does have a direction.

Erase the board and copy the following diagram.




Ask the participants to fill in the empty box. Continue the arrow road and ask, "What are some other numbers on this road?"

The exercises so far were not really problems. Rather, they were designed to introduce the language of arrows. Discuss with the participants the relationships that can be expressed with the arrows. The arrows can be used to model functions and relations. The language enables the graphic representation of the processes involved in comparing, analyzing, and operating on sets.

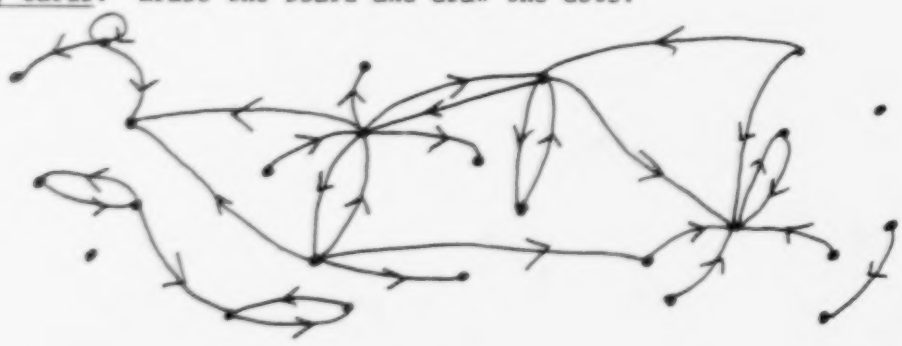
Now introduce one of the problem situations.

Classroom Friends. Erase the board and draw the following diagram on the chalkboard using blue chalk for the arrows:

Say, "This is a picture of students in a class. The blue arrow means 'You are a girl'". Ask participants "What can you say about the sex of these students?" Have participants come up to the chalkboard and label dots that represent girls with a G. Also have participants add any blue arrows that are definitely missing. For example, consider this part of the picture:

A and B are both girls. Therefore, two arrows can be drawn. C can say to B, "You are a girl," and D can say to B, "you are a girl." When all the definitely blue arrows are placed in the picture say, "There are no more blue arrows." Add a red arrow that means, "You are a boy." Have the participants add all the red arrows that they can.

Holiday Cards. Erase the board and draw the dots.

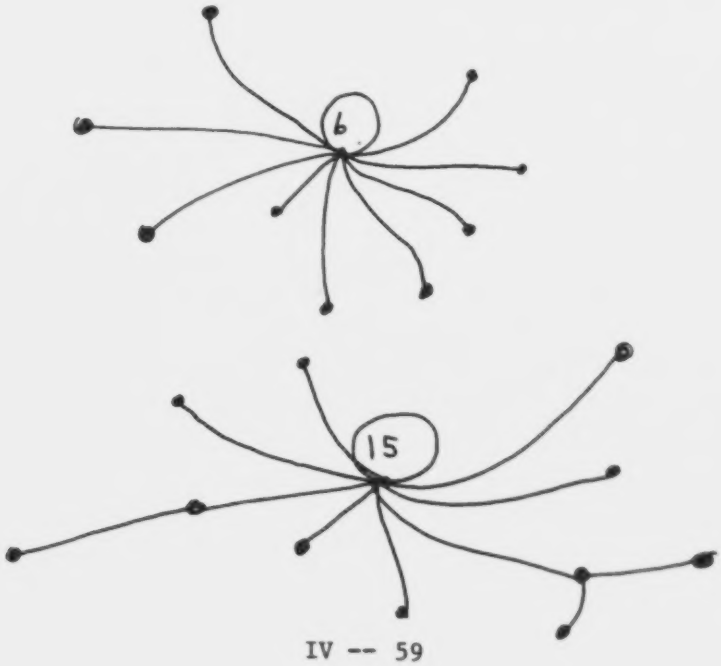


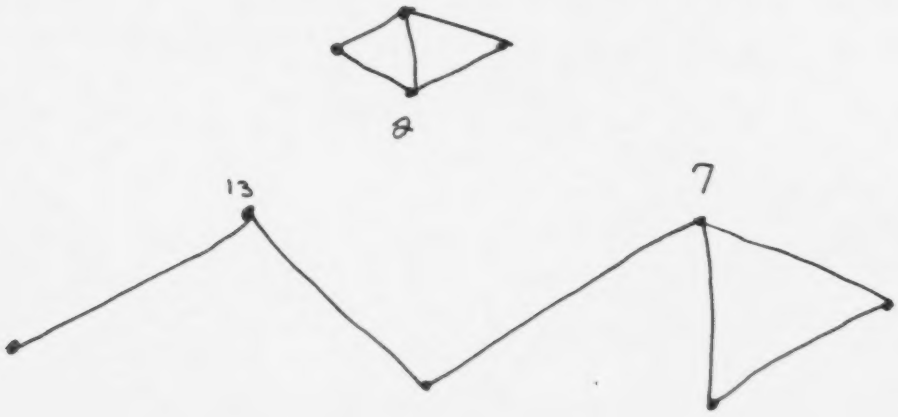
Introduce an arrow that means, "I sent you a holiday card." Draw in the arrows as shown in the picture. The following statements should be made by you. Pause for discussion between each statement.

> "This is a picture of my family - my brothers, sisters, parents, grandparents, cousins, uncles, and aunts. Can you figure out where I am in this picture?"
>
> "Cousin Ted sent a card to himself. Where is Ted? Label the dot."
>
> "My grandfather sent out the most cards and he sent me one. Where is my grandfather? Label the dot. Where might I be?"
>
> "Ted didn't send me a card. Where might I be?"
>
> "My mother received the most cards and I sent her one. Where is my mother? Label the dot. Where might I be?"
>
> "I sent out more cards then I received. Where am I?"

Ask a participant to make up some more statements to help the group figure out where another family member is in the picture.

Chatty Numbers. Erase the board and draw the four diagrams below on the board. Introduce the arrow expressed by the rule, "Two numbers can chat with each other, if and only if one number is a factor of the other." Ask participants to come to the board and label dots.






After all the dots are labeled, pose the problem:

> The rule does not allow 15 and 6 to chat. How could they send messages to each other?

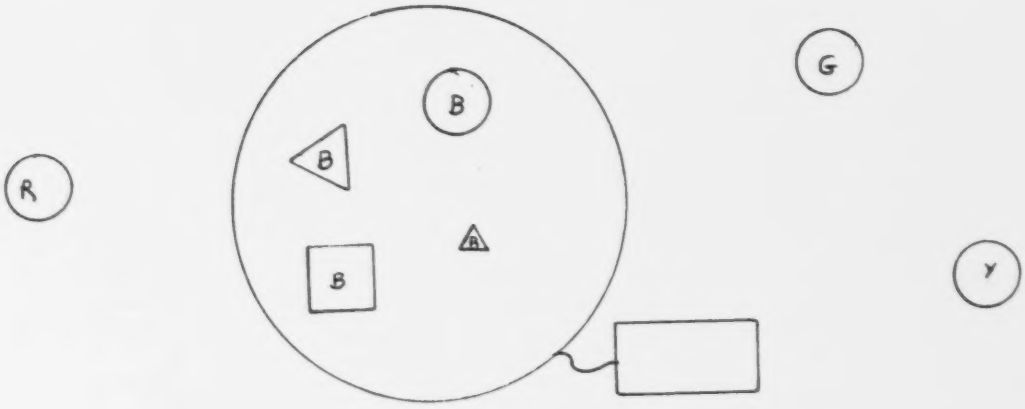
Lead the participants in discussing the solutions to this problem. You may want to ask the participants to make up their own problems with chatty numbers.

Discuss the language of arrows with the participants. Probe about the problem-solving strategies and skills used in doing these activities. Stress the versatility of the model for posing and solving problems. See TOPS (1982) and CSMP (1978) for more activities with arrows.

Activity IV. 4

Prepare to use the language of strings by cutting out a set of Attribute Blocks and Cards. If your chalkboard is magnetic, attach pieces of magnetic material to the back of your set of cut-out paper Attribute Blocks. If not, use pieces of masking tape to place the pieces on the board.

Introduce the string game by drawing a large circle on the chalkboard and placing an Attribute Card face down next to it as a label. Do not begin with a not-card (e.g., "Not Blue").




"This is a string. Inside the string I will place attribute blocks. Figure out the label for this string."

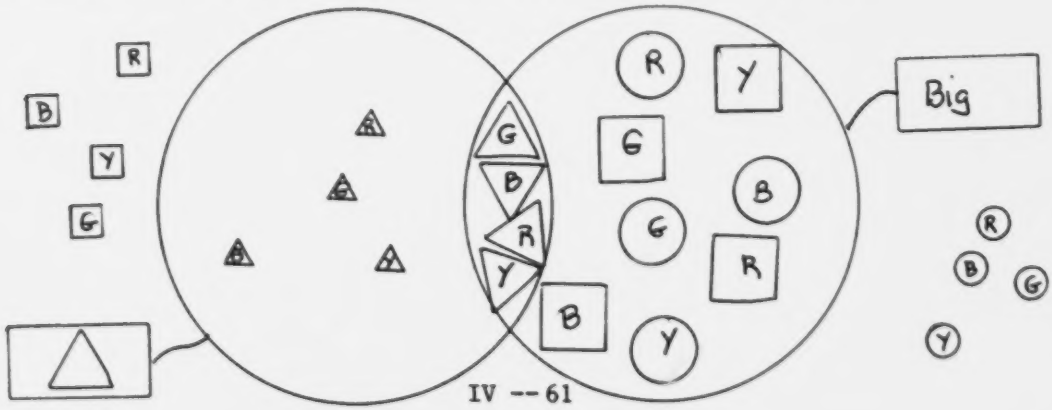
Place pieces inside the string that belong to the set, and pieces outside the string that don't belong. For example, if the hidden label says "Blue," place blue shapes in the string. Ask participants to show when they have figured out the label by telling you what pieces can go in the string. Do not ask them to verbalize their guess for the label until all the pieces are placed either inside or outside.

Now draw two strings on the chalkboard, one in red and one in blue chalk. Attach an Attribute Card face down next to each string. Divide the participants into two teams, and put half the Attribute Blocks on each side of the chalkboard for each team.

The object of the game is to place all the pieces on the chalkboard and identify the hidden labels. Only when a team has correctly placed all the pieces can the last player of that team make an identification of the labels.

Players should not talk. The team and the players on the teams take turns. You are the judge. You may want to prepare an answer sheet for the particular Attribute Cards you use to ensure accuracy of placement.

An example of a completed game with open labels is shown below:



Repeat the game several times, changing the labels. Use the labels such as "Not Blue" in subsequent games. You may choose to encourage talking between team members for these games.

Discuss the activity with the participants. Some questions you might raise are:

- What advantages and disadvantages are there to this competitive team approach to problem solving?
- What problem-solving strategies and skills were used?
- What are some possible variations on the string game?

Although you will want to point out that the language of strings is a model for classification concepts, the focus of the discussion should be on the usefulness of the language for posing and solving a multitude of problems. TOPS (1982) and CSMP (1978) contain numerous suggestions of problems with the language of strings.

## USING NEW IDEAS

### Summary Activity

The purpose of this last activity is to help the participants think about some ways that the training has affected or will affect their teaching. In addition, they are asked to identify the assistance they need to implement the ideas presented in the four workshops.

Allow five to ten minutes for the participants to respond to the questions in writing, and then, encourage them to share their responses orally with the group. You may also want to collect their responses to examine them in more depth. The last section in this manual will include follow-up suggestions and ways to support the implementation efforts of the participants. You should share your future plans at the end of the workshop.

Training Materials for Module IV

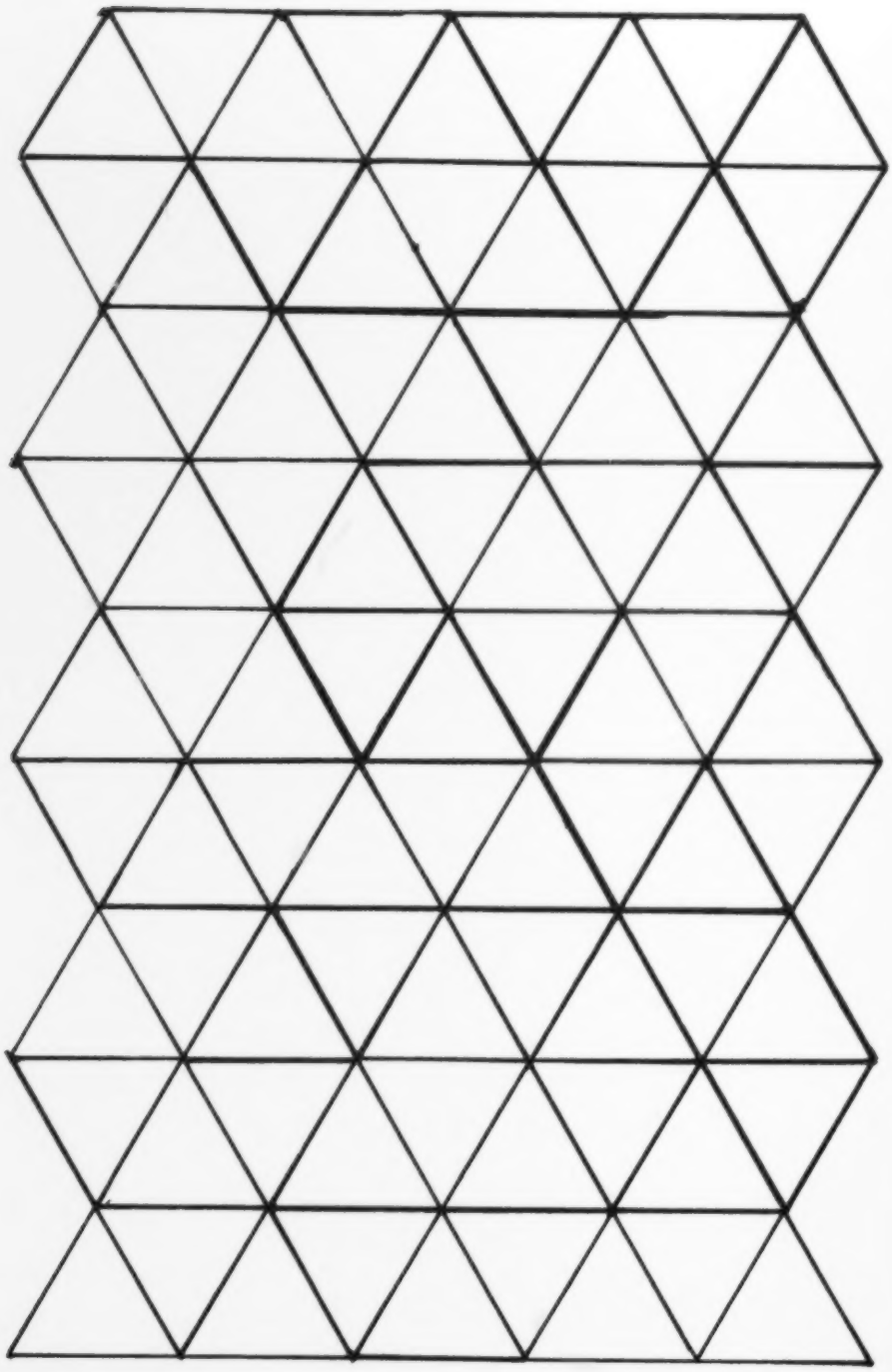
## OBJECTIVES FOR WORKSHOP IV

The objectives for this workshop are:

1. To explore problem-solving activities with manipulative materials; and
2. To examine the use of mathematical models to pose and solve problems.

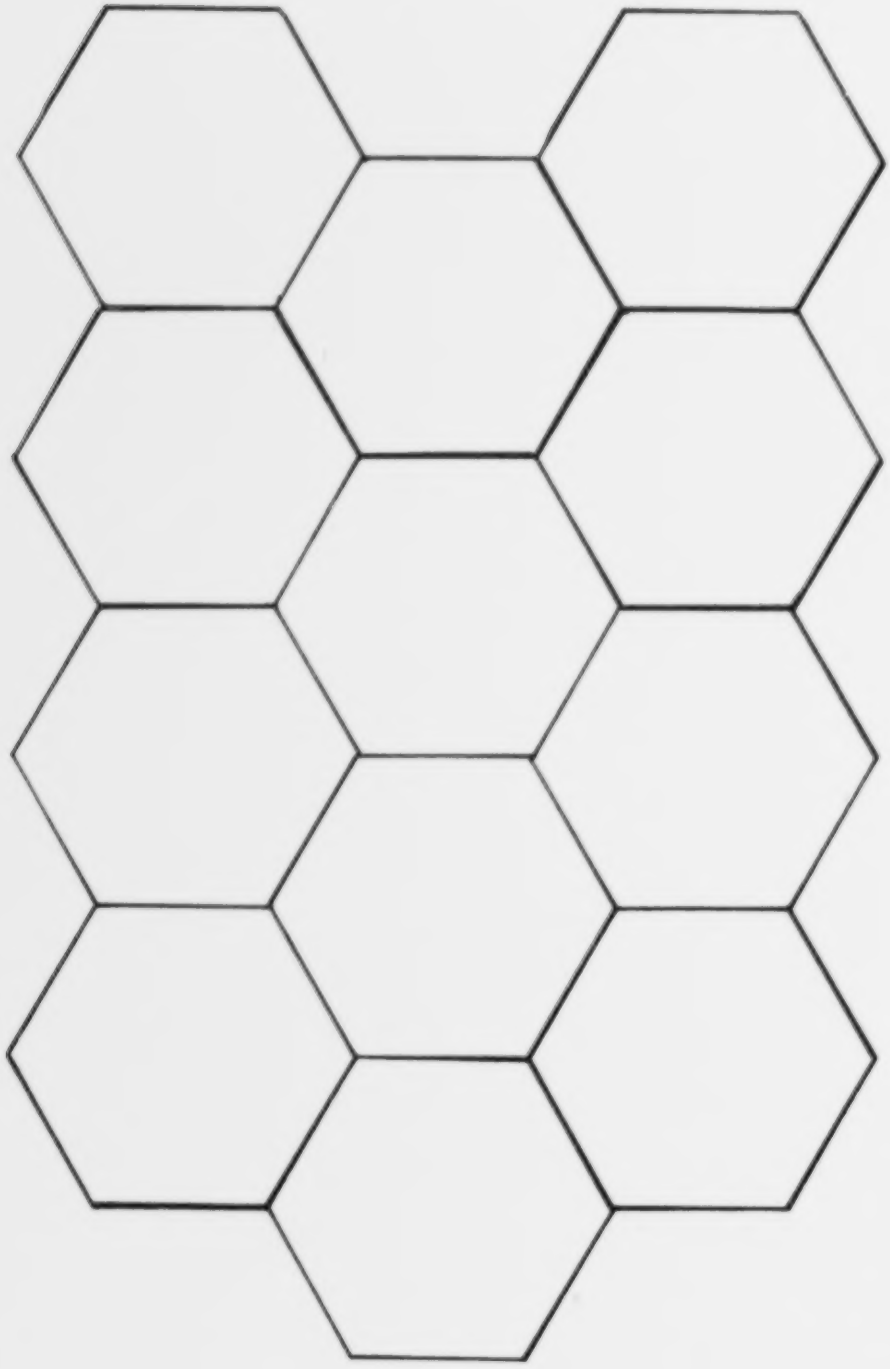
PATTERN BLOCKS

Green Triangles



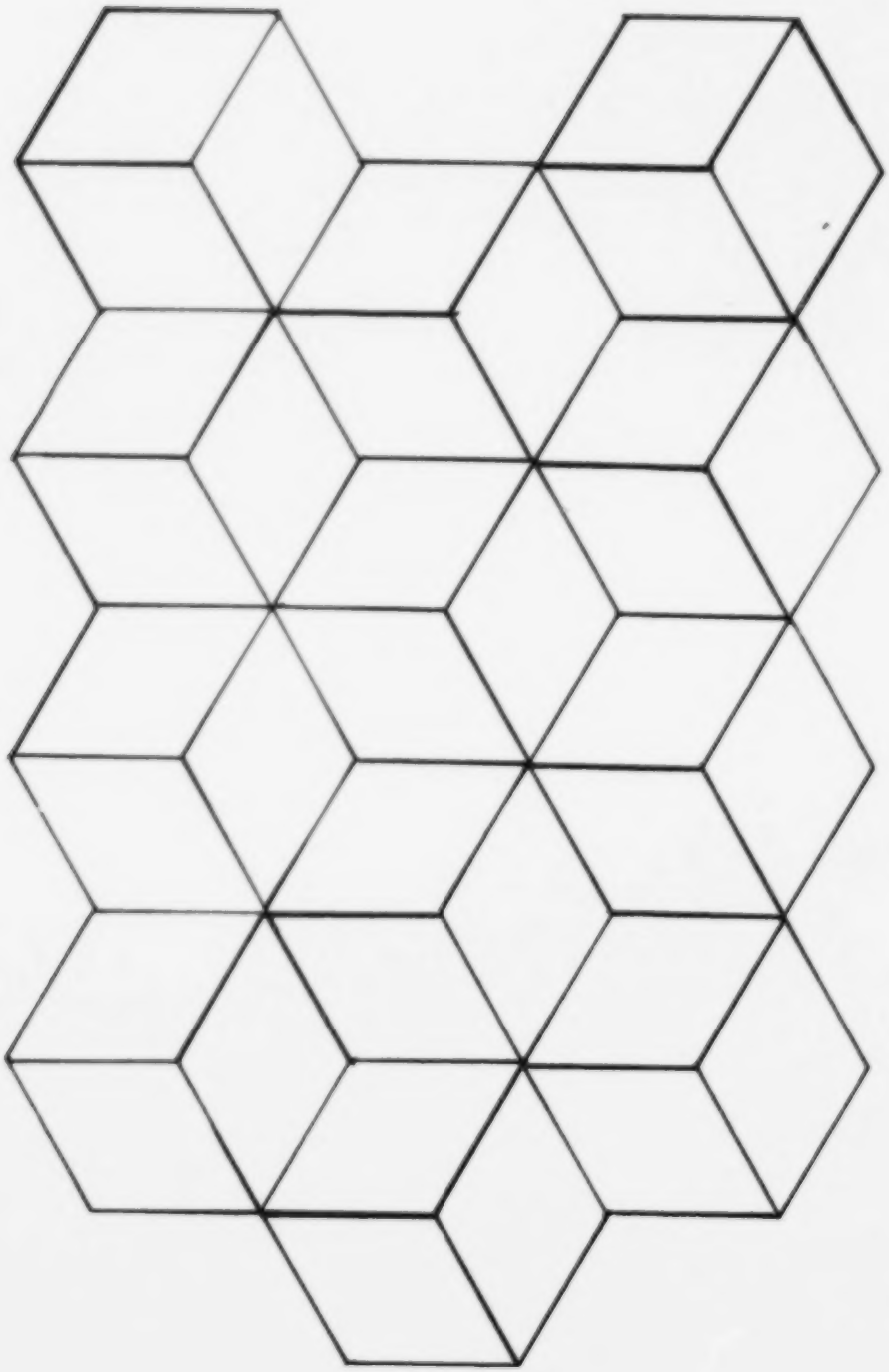
PATTERN BLOCKS

Yellow Hexagons



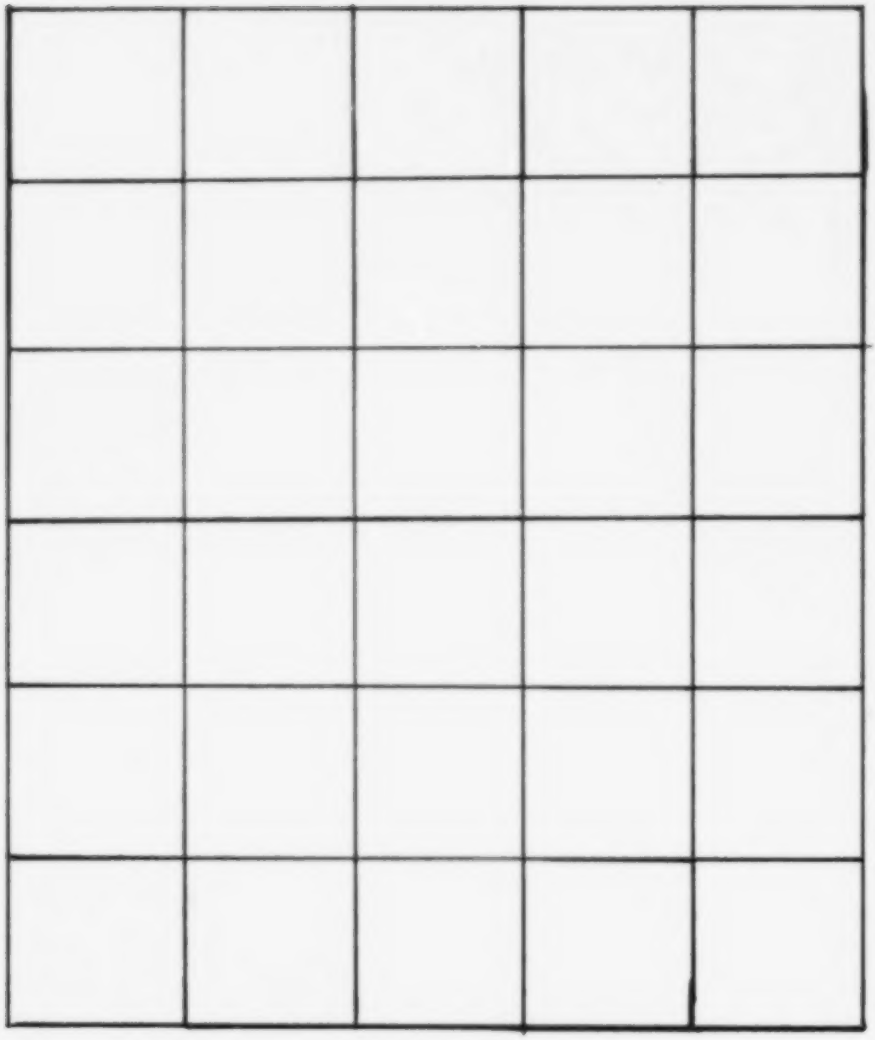
PATTERN BLOCKS

Blue Rhombi



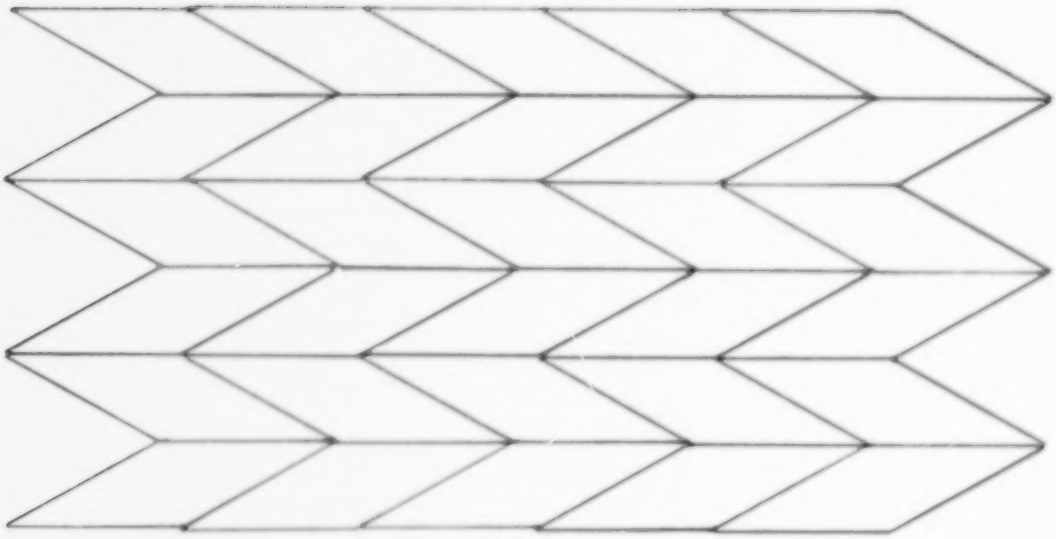
PATTERN BLOCKS

Orange Squares



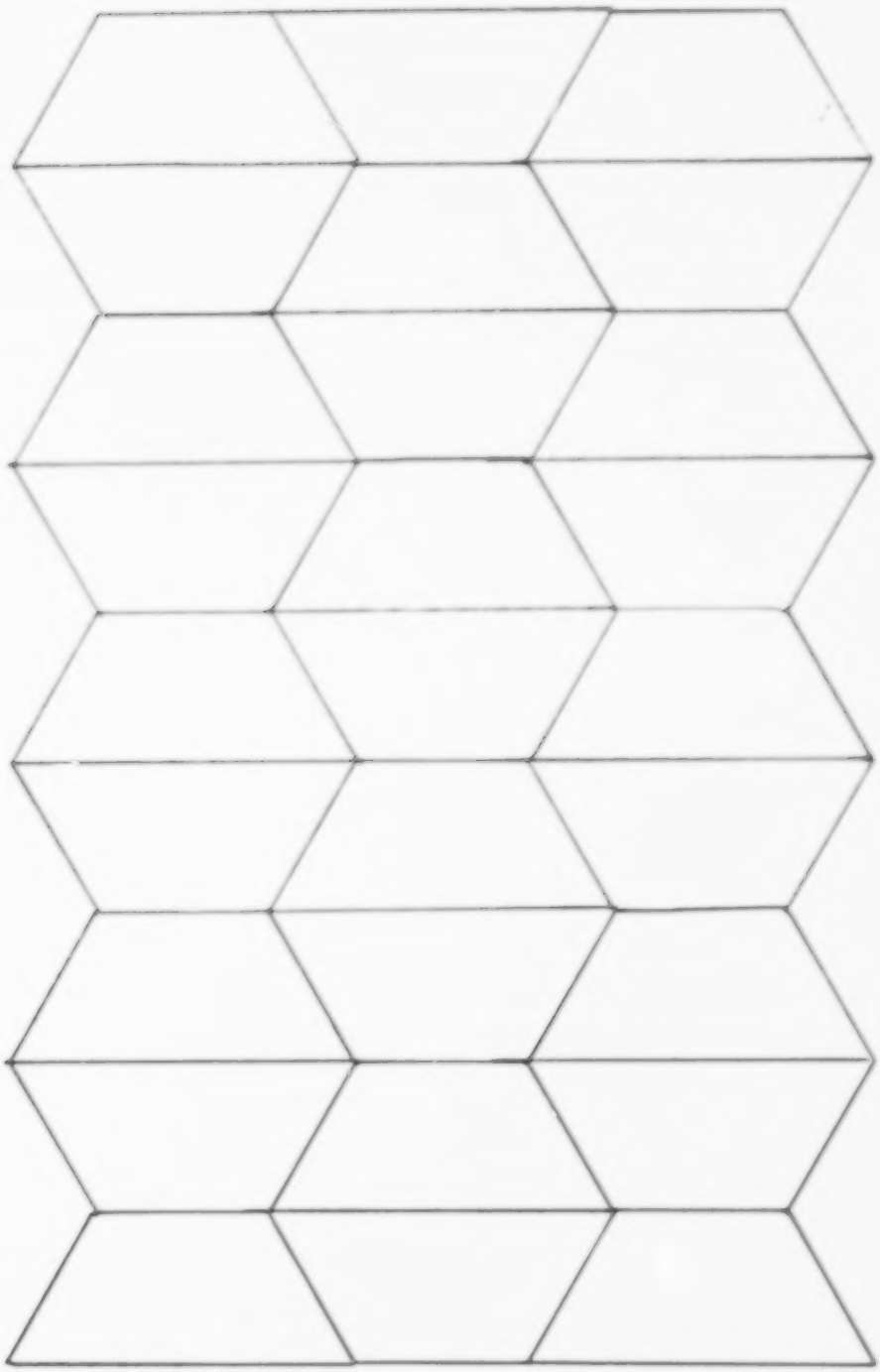
PATTERN BLOCKS

Tan Parallelograms

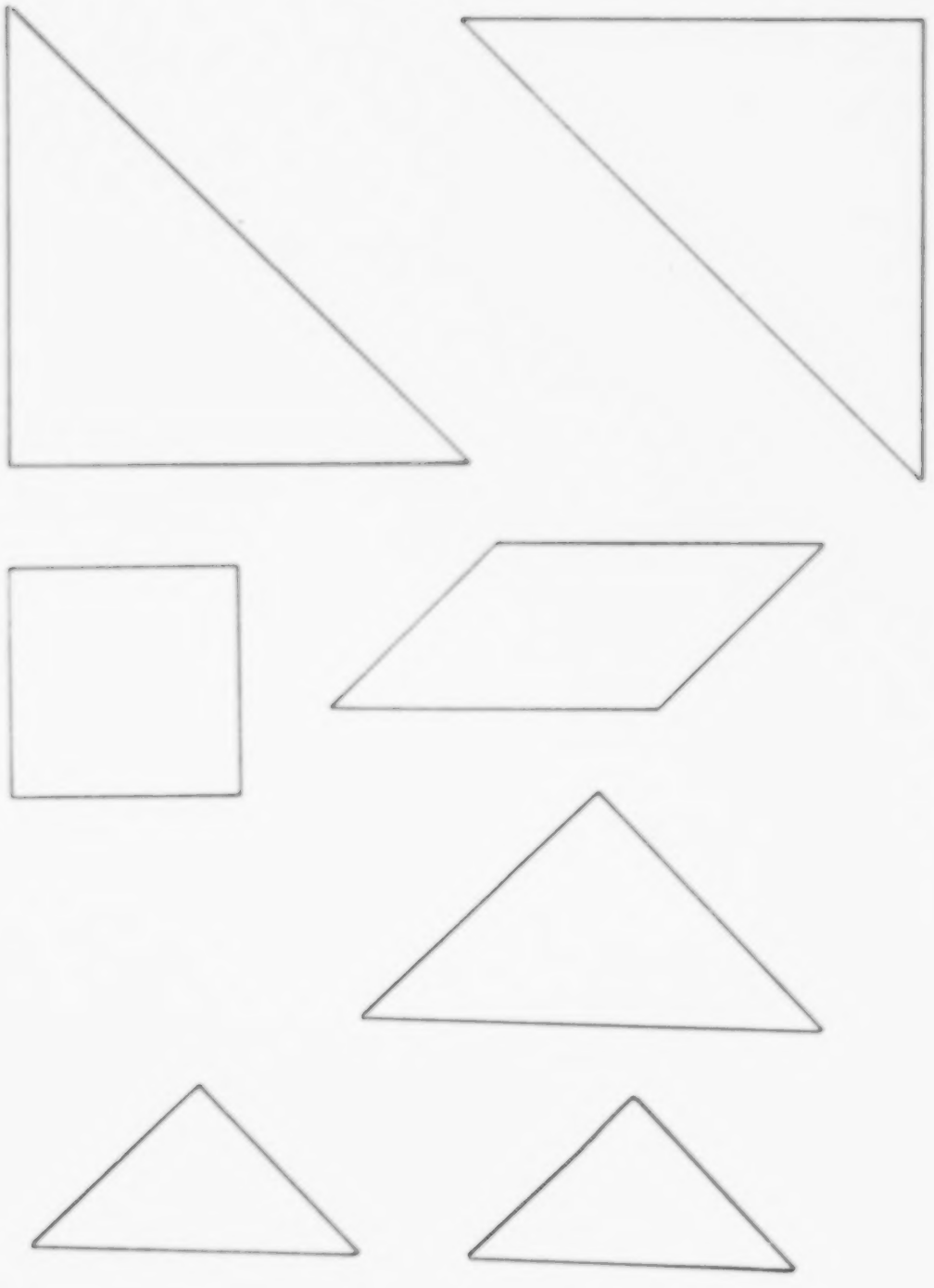


PATTERN BLOCKS

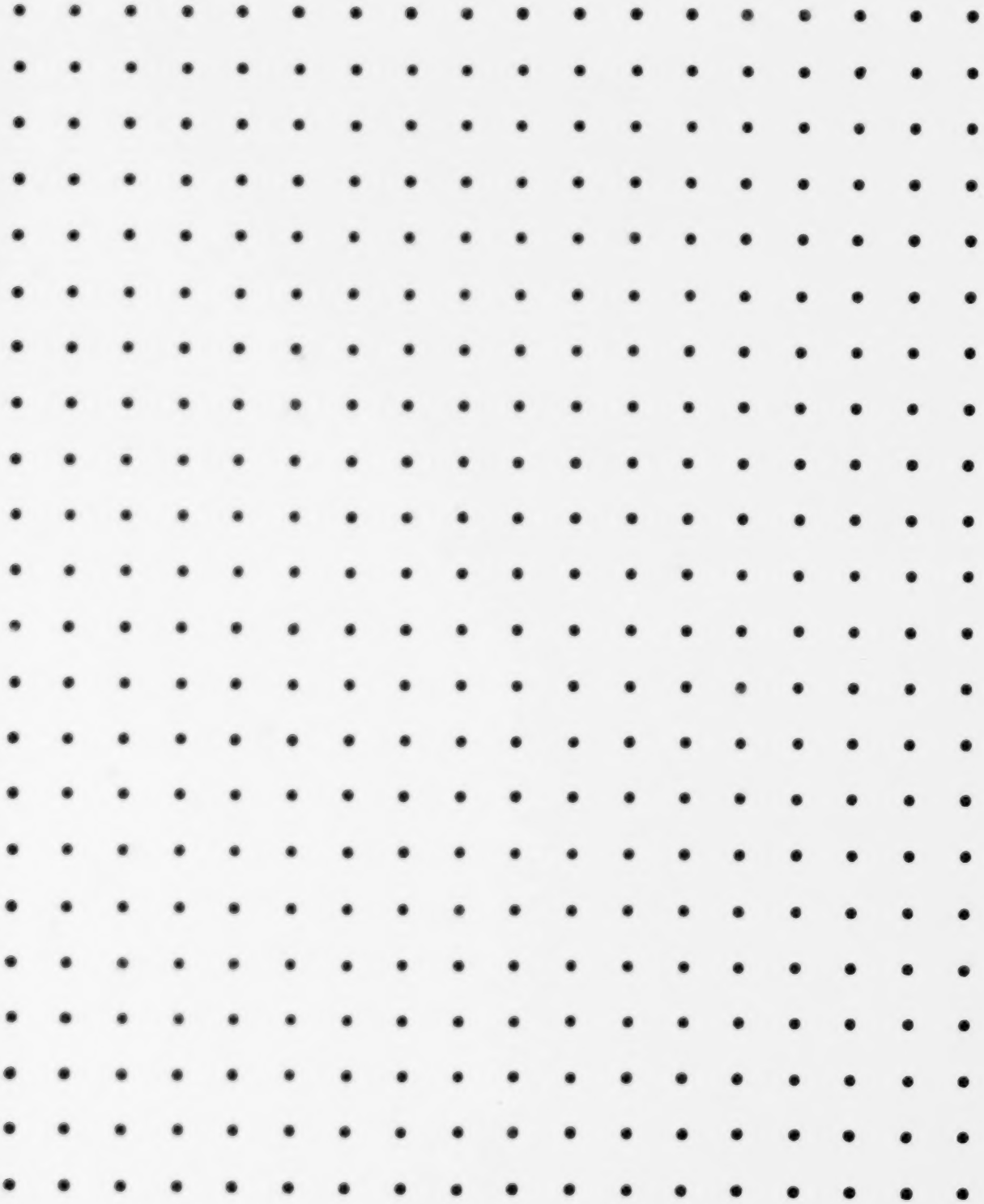
Red Trapezpoids



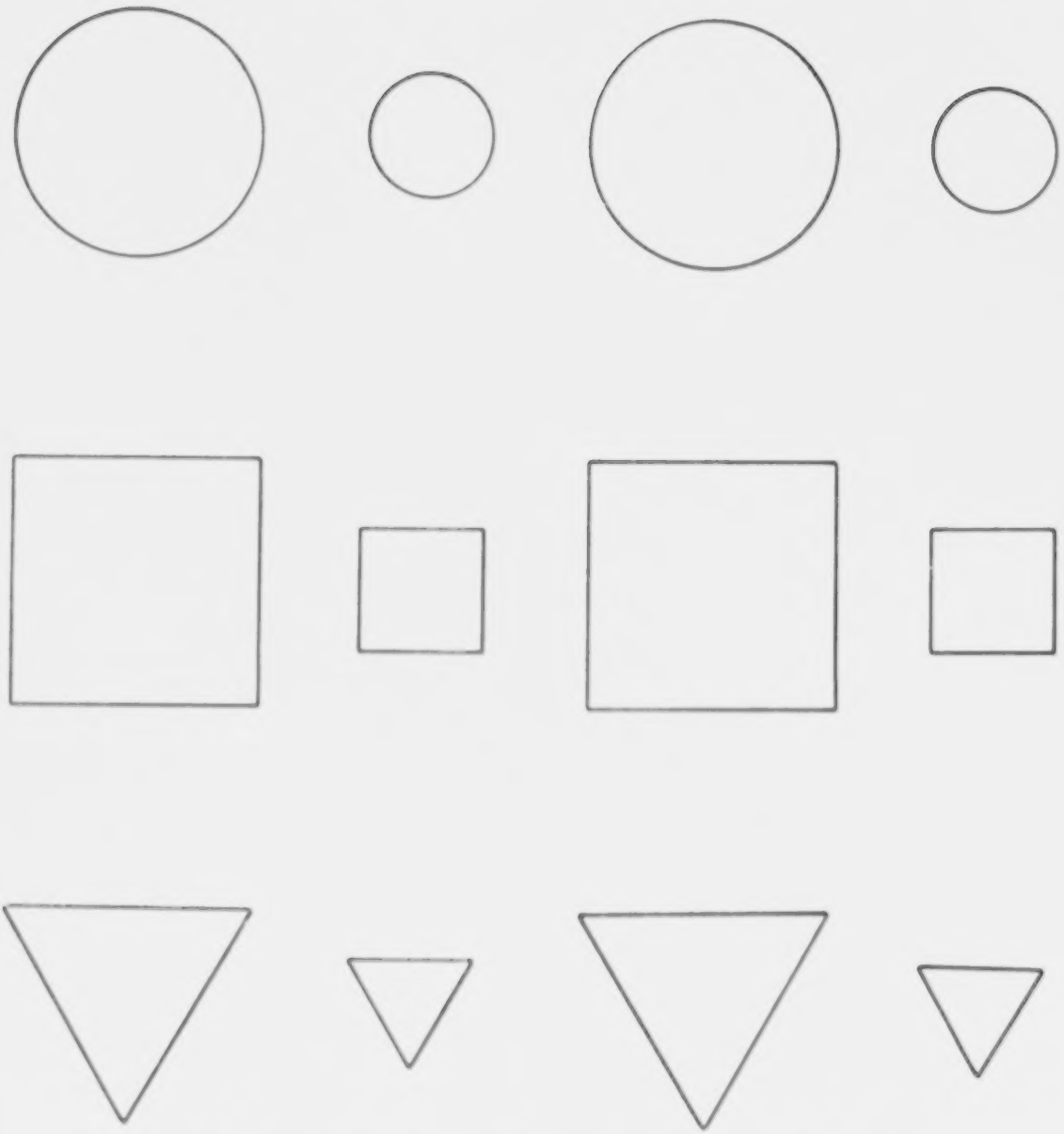
TANGRAM PIECES



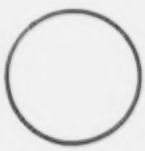
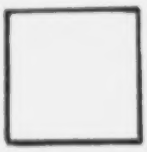
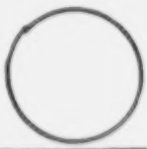
DOT PAPER



ATTRIBUTE BLOCKS



ATTRIBUTE CARDS

| | | | |
|---|---|---|---|
| RED | YELLOW | GREEN | BLUE |
| NOT RED | NOT YELLOW | NOT GREEN | NOT BLUE |
|  | |  | BIG |
| NOT  | NOT | NOT | LITTLE |